CONNECTION MATRICES IN COMBINATORIAL TOPOLOGICAL DYNAMICS

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Abstract. Connection matrices are one of the central tools in Conley's approach to the study of dynamical systems, as they provide information on the existence of connecting orbits in Morse decompositions. They may be considered a generalization of the boundary operator in the Morse complex in Morse theory. Their computability has recently been addressed by Harker, Mischaikow, and Spendlove [8] in the context of lattice filtered chain complexes. In the current paper, we extend the recently introduced Conley theory for combinatorial vector and multivector fields on Lefschetz complexes [13] by transferring the concept of connection matrix to this setting. This is accomplished by the notion of connection matrix for arbitrary poset filtered chain complexes, as well as an associated equivalence, which allows for changes in the underlying posets. We show that for the special case of gradient combinatorial vector fields in the sense of Forman [6], connection matrices are necessarily unique. Thus, the classical results of Reineck [20, 21] have a natural analogue in the combinatorial setting.

Contents

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1. Introduction

Classical Morse theory concerns a compact smooth manifold together with a smooth, real-valued function with non-degenerate critical points. An important part of the theory concerns the Morse complex which is a chain complex whose ith chain group is a free abelian group spanned by critical points of Morse index i and whose boundary homomorphism is defined by counting the (oriented) flow lines between critical points in the gradient flow induced by the Morse function (see [12, Section 4.2]). The fundamental result of classical Morse theory states that the homology of the manifold is isomorphic to the homology of the Morse complex.

The stationary points of the gradient flow in Morse theory, which are precisely the critical points of the Morse function, provide the simplest example of an isolated invariant set, a key concept of Conley theory [3]. For every isolated invariant set there is a homology module associated with it. It is called the homology Conley index. A (minimal) Morse decomposition is a decomposition of space into a partially ordered collection of isolated invariant sets, called Morse sets, such that every recurrent trajectory (in particular every stationary or periodic trajectory) is located in a Morse set and every nonrecurrent trajectory is a heteroclinic connection between Morse sets from a higher Morse set to a lower Morse set in the poset structure of the Morse decomposition. The collection of stationary points of the gradient flow of a Morse function provides the simplest example of a Morse decomposition in which the Morse sets are just the stationary points and the Conley index of a stationary point coincides with the homology of a pointed k-dimensional sphere with k equal to the Morse index of the point.

Conley theory in its simplest form may be viewed as a twofold generalization of Morse theory. On one hand it substantially weakens the general assumptions by replacing the smooth manifold by a compact metric space and the gradient flow of the Morse function by an arbitrary (semi)flow. On the other hand it replaces the collection of critical points of the Morse function by the more general Morse decomposition in which the counterpart of the Morse complex takes the form of the direct sum of the Conley indexes of all Morse sets. The homology of this generalized complex, as in the Morse theory, is isomorphic to the homology of the space. The boundary operator in this setting is called the connection matrix [7]. As in the Morse theory the connection matrix is defined in terms of the flow. The construction of connection matrix is technically complicated, in part because the generalized setting captures the situations of bifurcations when, unlike the Morse theory, the connection matrix need not be uniquely determined by the flow. Since a few years ago, connection matrices, despite their complicated construction, may be computed by algorithmic means due to an efficient algorithm recently proposed by Harker, Mischaikow, and Spendlove [8]. The algorithm utilizes a clear separation of dynamics and algebra which facilitates viewing a connection matrix as a purely algebraic object associated with every lattice filtered chain complex and considering a connection matrix of a Morse decomposition as a connection matrix of a filtered chain complex constructed from a lattice of attractors of the Morse decomposition.

Conley theory, in particular via connection matrices, is a very useful tool in the qualitative study of dynamical systems. However, to apply it one requires a well-defined dynamical system on a compact metric space. This is not the case when the dynamical system is exposed only via a finite set of samples as in the case of time series collected from observations or experiments. The study of dynamical systems known only from samples becomes an important part of the rapidly growing field of data science. In this context a generalization of Morse theory presented by Robin Forman [6] turned out to be very fruitful. In this generalization the smooth manifold is replaced by a finite CW complex and the gradient vector field of the Morse function by the concept of a combinatorial vector field. These structures may be easily constructed from data and analyzed by means of the combinatorial, also called discrete, Morse theory by Forman.

Recently, the concepts of isolated invariant set and Conley index have been carried over to this combinatorial setting [13, 17, 11]. In this paper we extend the ideas introduced there by constructing connection matrices of a Morse decomposition of a combinatorial multivector field. A problem here is that the dynamics of combinatorial vector and multivector fields is, by its very nature, multivalued. One can give examples which demonstrate that index filtrations, that is certain lattice homomorphisms needed in the construction of the connection matrices, fail to exist in the case of multivalued dynamics. Actually, such examples exist also in the setting of multivalued flows in compact metric spaces. We overcome this difficulty by looking into the whole family of Morse decompositions with varying poset structure. By adding some Morse sets with trivial Conley index we reduce the problem to a special case when the index filtrations do exist and their construction is straightforward. The added Morse sets may then be removed by introducing a certain equivalence in the category of filtered chain complexes with a varying poset structure. This lets us consider connection matrices in the setting of combinatorial vector and multivector fields and prove the uniqueness of connection matrices for gradient combinatorial vector fields.

2. Main results

The main result of the paper is the following theorem.

Theorem 2.1. *Assume $\mathcal{V}$ is a gradient combinatorial vector field on a Lefschetz complex X. Then the Morse decomposition consisting of all the critical cells of $\mathcal{V}$ has precisely one connection matrix. It coincides with the matrix of the boundary operator of the associated Morse complex.*

The connection matrix need not be unique in the case of a general combinatorial multivector field or in the case of a non-gradient combinatorial

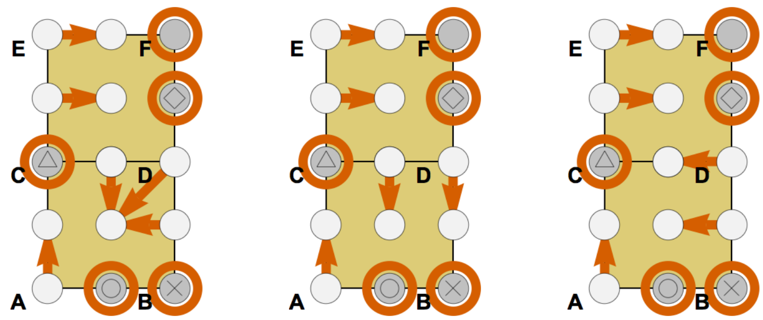

FIGURE 1. A multivector field (left) and its two different Forman refinements (middle and right).

vector field. This is illustrated in Figure 1 with three examples: a combinatorial multivector field in the left and two combinatorial vector fields, one in the middle and in the right. All three examples have the same collection of critical cells $\mathcal{M} := \{\, \mathbf{B}, \mathbf{C}, \mathbf{F}, \mathbf{AB}, \mathbf{DF} \,\}$ and $\mathcal{M}$ is a Morse decomposition for all of them. One can verify that the left example has two connection matrices with coefficients in $\mathbb{Z}_2$:

$$\mathcal{C}_1 := \begin{array}{c|ccccc} & \mathbf{B} & \mathbf{C} & \mathbf{F} & \mathbf{AB} & \mathbf{DF} \\ \hline \mathbf{B} & 0 & 0 & 0 & 1 & 1 \\ \mathbf{C} & 0 & 0 & 0 & 1 & 0 \\ \mathbf{F} & 0 & 0 & 0 & 0 & 1 \\ \mathbf{AB} & 0 & 0 & 0 & 0 & 0 \\ \mathbf{DF} & 0 & 0 & 0 & 0 & 0 \end{array} \;, \quad \mathcal{C}_2 := \begin{array}{c|ccccc} & \mathbf{B} & \mathbf{C} & \mathbf{F} & \mathbf{AB} & \mathbf{DF} \\ \hline \mathbf{B} & 0 & 0 & 0 & 1 & 0 \\ \mathbf{C} & 0 & 0 & 0 & 1 & 1 \\ \mathbf{F} & 0 & 0 & 0 & 0 & 1 \\ \mathbf{AB} & 0 & 0 & 0 & 0 & 0 \\ \mathbf{DF} & 0 & 0 & 0 & 0 & 0 \end{array}$$

The matrix $\mathcal{C}_1$ is the unique matrix of the combinatorial multivector field shown in the middle of Figure 1, while $\mathcal{C}_2$ is the unique matrix for the combinatorial multivector field shown on the right.

3. Preliminaries

3.1. **Sets and maps.** We denote the sets of reals, integers, positive integers, and non-negative integers by $\mathbb{R}$, $\mathbb{Z}$, $\mathbb{N}$, and $\mathbb{N}_0$, respectively. Given a set X, we write $\operatorname{card} X$ for the number of elements of X and we denote the family of all subsets of X by $\mathcal{P}(X)$. By a *partition* of a set X we mean a family $\mathcal{E} \subset \mathcal{P}(X)$ of mutually disjoint non-empty subsets of X such that $\bigcup \mathcal{E} = X$. For a subfamily $\mathcal{E}' \subset \mathcal{E}$ of a partition $\mathcal{E}$ of X we will use the compact notation $|\mathcal{E}'|$ to denote the union of sets in $\mathcal{E}'$, that is, $|\mathcal{E}'| = \bigcup \mathcal{E}'$. We say that a subset $E \subset X$ is *$\mathcal{E}$-compatible* with respect to a partition $\mathcal{E}$ of X if we have $E = |\mathcal{E}'|$ for a subfamily $\mathcal{E}' \subset \mathcal{E}$.

A *K-indexed family* of subsets of a set X is an injective map $K \ni k \mapsto X_k \in \mathcal{P}(X)$. We denote such a family by $(X_k)_{k\in K}$. Note that every family $\mathcal{E} \subset \mathcal{P}(X)$ may be considered as an $\mathcal{E}$-indexed family $\mathcal{E} \ni E \mapsto E \in \mathcal{P}(X)$. In this case we say that $\mathcal{E}$ is a *self-indexing* family. By a *K-gradation* of a set X we mean a K-indexed family $(X_k)_{k\in K}$ such that $\{X_k\}_{k\in K}$ is a partition of X. Note that every surjective map $f : X \to K$ provides a K-gradation $(f^{-1}(k))_{k\in K}$ of X. We refer to this K-gradation as the *f-gradation of X*. Given $A \subset X$ and a gradation $(X_k)_{k\in K}$ of X, by the *induced gradation* we mean the gradation $(A\cap X_k)_{k\in K_A}$ of A where $K_A := \{\, k \in K \mid A \cap X_k \neq \varnothing \,\}$.

We write $f : X \nrightarrow Y$ for a partial map from X to Y, that is, a map defined on a subset $\operatorname{dom} f \subset X$, called the *domain* of f, and such that the set of values of f, denoted $\operatorname{im} f$, is contained in Y. Partial maps are composed in the obvious way: If $f : X \nrightarrow Y$ and $g : Y \nrightarrow Z$ are partial maps, then the domain of their composition $g \circ f : X \nrightarrow Z$ is given by $\operatorname{dom}(g \circ f) = f^{-1}(\operatorname{dom} g)$, and on this domain we have $(g \circ f)(x) = g(f(x))$.

In the sequel, we work with the *category of finite sets with a distinguished subset*, denoted DSet, and defined as follows. The objects of DSet are pairs $(X, X_\star)$, where X is a finite set and $X_\star \subset X$ is a distinguished subset. The morphisms from $(X, X_\star)$ to $(Y, Y_\star)$ in DSet are *subset preserving partial maps*, that is, partial maps $f : X \nrightarrow Y$ such that $X_\star \subset \operatorname{dom} f$ and $f(X_\star) \subset Y_\star$. One easily verifies that DSet with the composition of morphisms defined as the composition of partial maps and the identity morphism defined as the identity map is indeed a category.

3.2. **Relations, multivalued maps and digraphs.** Given a set X and a binary relation $R \subset X \times X$, we use the shorthand xRy for $(x, y) \in R$. The *inverse* of R, denoted R^{-1}, is the relation $R^{-1} := \{\, (y, x) \mid xRy \,\}$. By the *transitive closure* of R we mean the relation $\bar{R} \subset X \times X$ given by $x\bar{R}y$ if there exists a sequence $x = x_0, x_1, \ldots, x_n = y$ such that $n \geq 1$ and $x_{i-1}Rx_i$ for $i = 1, 2, \ldots, n$. Note that $\bar{R}$ is transitive but need not be reflexive. The relation $\bar{R}\cup \operatorname{id}_X$, where id_X stands for the identity relation on X, is reflexive and transitive. Hence, it is a *preorder*, called the *preorder induced by R*. An element $y \in X$ *covers* an $x \in X$ in the relation R if xRy and $x \neq y$, but there is no $z \in X$ such that $x \neq z \neq y$ and both xRz and zRy are satisfied.

A multivalued map $F : X \multimap Y$ is a map $F : X \to \mathcal{P}(Y)$. For $A \subset X$ we define the *image of A* by $F(A) := \bigcup \{\, F(x) \mid x \in A \,\}$ and for $B \subset Y$ we define the *preimage of B* by $F^{-1}(B) := \{\, x \in X \mid F(x) \cap B \neq \varnothing \,\}$.

Given a relation R, we associate with it a multivalued map $F_R : X \multimap X$, by $F_R(x) := R(x)$, where $R(x) := \{\, y \in X \mid xRy \,\}$ is the *image of $x \in X$ in R*. Obviously, $R \mapsto F_R$ is a one-to-one correspondence between binary relations in X and multivalued maps from X to X. Often, it will be convenient to interpret the relation R as a directed graph whose set of vertices is X and a directed arrow goes from x to y whenever xRy. The three concepts relation, multivalued map and directed graph are equivalent on the formal level and the distinction is used only to emphasize different viewpoints. However, in this paper it will be convenient to use all these concepts interchangeably.

3.3. **Partial orders.** Recall that a *preorder* in a set P is a reflexive and transitive relation in P. If not stated otherwise we denote a preorder by $\leq$ and its inverse by $\geq$. We also write $<$ and $>$ for the associated strict preorders, that is, the relations $\leq$ and $\geq$ excluding equality. A *partial order* is a preorder which is antisymmetric. We also recall that a *preordered set* (a *poset*) is a pair $(P, \leq)$ where P is a set and $\leq$ is a preorder (a partial order) in P. Whenever the preorder (the partial order) is clear from the context, we shorten the notation $(P, \leq)$ to P.

Given a preordered set P we say that a $q \in P$ *covers* a $p \in P$ if $p < q$ and there is no $r \in P$ such that $p < r < q$. We say that p is a *predecessor* of q if q covers p. Recall that a subset $A \subset P$ is *convex* if the conditions $x, z \in A$ and $x \leq y \leq z$ imply $y \in A$. A set $A \subset P$ is a *down set* or a *lower set* if the conditions $z \in A$ and $y \leq z$ yield $y \in A$. Dually, the subset $A \subset P$ is an *upper set* if $z \in A$ and $z \leq y$ guarantee $y \in A$.

We denote the family of all down sets in P by $\operatorname{Down}(P)$. For $A \subset P$ we further write $A^{\leq} := \{\, x \in P \mid \exists_{a \in A}\ x \leq a \,\}$ and $A^{<} := A^{\leq} \setminus A$.

Proposition 3.1. *If I is convex, then both $I^{\leq}$ and $I^{<}$ are down sets.*

Proof: The verification that $I^{\leq}$ is a down set is straightforward. To see that $I^{<}$ is a down set take an $x \in I^{<}$. Then we have $x \notin I$ and $x < z$ for some $z \in I$. Let $y \leq x$. Then $y \in I^{\leq}$. Since I is convex, we cannot have the inclusion $y \in I$. It follows that $y \in I^{<}$. $\square$

Let P and P' be posets. We say that a partial map $f : P \nrightarrow P'$ is *order preserving* if the conditions $x, y \in \operatorname{dom} f$ and $x \leq y$ imply $f(x) \leq f(y)$. We say that f is an *order isomorphism* if f is an order preserving bijection such that f^{-1} is also order preserving.

We define the category DPSet of posets with a distinguished subset as follows. Its objects are pairs $(P, P_\star)$ where P is a finite poset and $P_\star \subset P$ is a distinguished subset. A morphism from $(P, P_\star)$ to $(P', P'_\star)$ in DPSet is an order preserving partial map $f : P \nrightarrow P'$ such that $P_\star \subset \operatorname{dom} f$ and $f(P_\star) \subset P'_\star$. One easily verifies that DPSet with the composition of morphisms defined as the composition of partial maps and the identity morphism defined as the identity map is indeed a category. Moreover, since every set may be considered as a poset partially ordered by the identity and, clearly, every partial map preserves identities, we may consider DSet as a subcategory of DPSet. In this case, every partial map is automatically order preserving.

To simplify notation, in the sequel we denote objects of DSet and DPSet with a single capital letter and the distinguished subset by the same letter with subscript $\star$.

3.4. **Distributive lattices.** For the terminology of this paper concerning lattices we refer to [23]. Here we briefly recall that a *lattice* is a triple $(\mathcal{L}, \vee, \wedge)$ where $\mathcal{L}$ is a set and $\vee, \wedge : \mathcal{L} \times \mathcal{L} \to \mathcal{L}$ are binary operations on $\mathcal{L}$ which are called *join* and *meet*, respectively, and which are idempotent, commutative, associative and satisfy the absorption rule, that is, $a \vee (a \wedge b) = a \wedge (a \vee b) = a$ for all $a, b \in \mathcal{L}$. A lattice is *distributive* if the operations $\vee$ and $\wedge$ are mutually distributive. If there exists an element $a \in \mathcal{L}$ such that $a = a \wedge b$ for every $b \in \mathcal{L}$, it is called *a zero element*. It is straightforward to observe that if a zero element exists, it is unique. We denote it by 0. Similarly, if there exists an element $a \in \mathcal{L}$ such that $a = a \vee b$ for every $b \in \mathcal{L}$, it is called *a unit element*. If a unit element exists, it is unique. We denote it by 1. A *bounded lattice* is a quintuple $(\mathcal{L}, 0, 1, \vee, \wedge)$ such that $(\mathcal{L}, \vee, \wedge)$ is a lattice with $0 \in \mathcal{L}$ as zero element and $1 \in \mathcal{L}$ as unit element.

In the sequel we assume that every considered lattice is a finite, distributive, and bounded lattice. By saying that $\mathcal{L}$ is a lattice, we actually mean the lattice $(\mathcal{L}, 0, 1, \vee, \wedge)$ in general notation — unless we specify precisely or it is clear from context what the zero, the unit, and the operators $\vee$ and $\wedge$ are.

A subset $\mathcal{L}' \subset \mathcal{L}$ is a *sublattice* of a lattice $\mathcal{L}$ if the zero and unit elements of $\mathcal{L}$ belong to $\mathcal{L}'$ and the binary operations $\vee$ and $\wedge$ in $\mathcal{L}$ are closed with respect to $\mathcal{L}'$, that is, we have both $\vee(\mathcal{L}' \times \mathcal{L}') \subset \mathcal{L}'$ and $\wedge(\mathcal{L}' \times \mathcal{L}') \subset \mathcal{L}'$. Then $\mathcal{L}'$ with 0, 1, and the restricted operations $\vee_{|\mathcal{L}' \times \mathcal{L}'}$, $\wedge_{|\mathcal{L}' \times \mathcal{L}'}$ is itself a lattice.

Given two lattices $\mathcal{L}$ and $\mathcal{L}'$, a map $h : \mathcal{L} \to \mathcal{L}'$ is called a *lattice homomorphism* (or briefly, a homomorphism) if it preserves the binary operations $\vee$ and $\wedge$, and it maps the zero and unit elements in $\mathcal{L}$ to the zero and unit elements in $\mathcal{L}'$, respectively.

A non-zero element $a \in \mathcal{L}$ is called *join-irreducible*, if $a = a_1 \vee a_2$ implies that $a = a_1$ or $a = a_2$. We denote the set of join-irreducible elements of $\mathcal{L}$ by $\operatorname{JIrr}(\mathcal{L})$. Every lattice induces a partial order $\leq_{\mathcal{L}}$ on $\mathcal{L}$ given by $a \leq_{\mathcal{L}} b$ if one has $a = a \wedge b$. We note that zero is the only element in the resulting poset $(X, \leq_{\mathcal{L}})$ which has no predecessor. In fact, we have the following proposition.

Proposition 3.2. *Assume that $\mathcal{L}$ is a finite, bounded, distributive lattice and that $a \in \mathcal{L}$. Then a is join-irreducible if and only if a has exactly one predecessor with respect to the lattice induced partial order $\leq_{\mathcal{L}}$.*

Proof: Assume $a \in \mathcal{L}$ is join-irreducible. Then, $a \neq 0$. Hence, a has a predecessor. Suppose b_i for $i \in \{1, 2\}$ are two predecessors of a. Then $b_i <_{\mathcal{L}} a$ and $b_i = b_i \wedge a$, as well as $(b_1 \vee b_2) \wedge a = (b_1 \wedge a) \vee (b_2 \wedge a) = b_1 \vee b_2$. Thus, $b_1 \vee b_2 \leq_{\mathcal{L}} a$ and, by the absorption rule, $b_i \leq_{\mathcal{L}} b_1 \vee b_2$ for $i = 1, 2$. Since b_i is a predecessor of a, either $b_1 \vee b_2 = b_i$ or $b_1 \vee b_2 = a$. The latter is not possible, because a is join-irreducible. Hence $b_1 \vee b_2 = b_i$ for $i \in \{1, 2\}$. In consequence $b_1 = b_2$, which proves the uniqueness of the predecessor.

To prove the opposite implication, assume to the contrary that a has exactly one predecessor and that a is not join-irreducible. Then $a \neq 0$ and $a = a_1 \vee a_2$ for some $a_1, a_2 \in \mathcal{L}$ such that $a_1 \neq a \neq a_2$. By the absorption rule $a_i \leq_{\mathcal{L}} a$ for $i \in \{1, 2\}$. Let $\bar{a}_i$ be a predecessor of a such that $a_i \leq_{\mathcal{L}} \bar{a}_i$. It suffices to prove that $\bar{a}_1 \neq \bar{a}_2$, because then a has more than one predecessor. Thus, assume $\bar{a}_1 = \bar{a}_2$ and set $\bar{a} := \bar{a}_1 = \bar{a}_2$. Then $(a_1 \vee a_2) \wedge \bar{a} = (a_1 \wedge \bar{a}) \vee (a_2 \wedge \bar{a}) = a_1 \vee a_2$. It immediately follows that one then has $a = a_1 \vee a_2 \leq_{\mathcal{L}} \bar{a} <_{\mathcal{L}} a$, a contradiction proving that $\bar{a}_1 \neq \bar{a}_2$.

Therefore, a has at least two predecessors — which in turn contradicts the assumption that a has exactly one predecessor. $\square$

If $a \in \mathcal{L}$ is join-irreducible, we denote the unique predecessor of a with respect to $\leq_{\mathcal{L}}$ by $a^\star$.

We recall that by $\mathcal{P}(X)$ we denote the family of all subsets of a set X. It is straightforward to observe that given an arbitrary finite set X, the quintuple $(\mathcal{P}(X), \varnothing, X, \cup, \cap)$ is a lattice. Moreover, the lattice induced partial order on $\mathcal{P}(X)$ coincides with the inclusion relation $\subset$. If P is a finite poset, then $\operatorname{Down}(P)$ is a sublattice of the lattice $\mathcal{P}(P)$. The Birkhoff representation theorem [2] states that every finite, bounded, distributive lattice is isomorphic to a lattice of down sets. More precisely, we have the following theorem.

Theorem 3.3 (Birkhoff, 1937)**.** *Every finite, bounded, distributive lattice $\mathcal{L}$ is isomorphic to the lattice of down sets of the partially ordered set of join-irreducible elements of $\mathcal{L}$.* $\square$

3.5. **Topological spaces.** For our terminology concerning topological spaces we refer the reader to [4, 19]. Recall that a *topology* on a set X is a family $\mathcal{T}$ of subsets of X which is closed under finite intersections and arbitrary unions, and which satisfies $\varnothing, X \in \mathcal{T}$. The sets in $\mathcal{T}$ are called *open*. The *interior* of A, denoted $\operatorname{int} A$, is the union of all open subsets of A. A subset $A \subseteq X$ is *closed* if $X \setminus A$ is open. We denote by $\operatorname{Clsd}(X)$ the family of closed subsets of X. Note that $\operatorname{Clsd}(X)$ is a sublattice of the lattice $\mathcal{P}(X)$ of subsets of X. The *closure* of A, denoted $\operatorname{cl}_{\mathcal{T}} A$, or by $\operatorname{cl} A$ if $\mathcal{T}$ is clear from context, is the intersection of all closed supersets of A.

Given two topological spaces $(X, \mathcal{T})$ and $(X', \mathcal{T}')$ we say that $f : X \to X'$ is *continuous* if $U \in \mathcal{T}'$ implies $f^{-1}(U) \in \mathcal{T}$. If we want to emphasize the involved topologies we say that $f : (X, \mathcal{T}) \to (X', \mathcal{T}')$ is continuous.

Recall that a subset A of a topological space X is called *locally closed*, if every point $x \in A$ admits a neighborhood U in X such that $A \cap U$ is closed in U. Locally closed sets as well as $\operatorname{mo} A := \operatorname{cl} A \setminus A$, which we refer to as the *mouth* of A, are important in the sequel.

Proposition 3.4 (see Problem 2.7.1 in [4])**.** *Assume that A is a subset of a topological space X. Then the following conditions are pairwise equivalent.*

(i) *A is locally closed,*
(ii) *the mouth of A is closed in X,*
(iii) *A is a difference of two closed subsets of X,*
(iv) *A is an intersection of an open set in X and a closed set in X.*

Proof: It is obvious that (ii) implies (iii), (iii) implies (iv), and (iv) implies (i). Thus, it suffices to prove that if (ii) fails, then so does (i). Hence, assume that the mouth $\operatorname{mo} A = \operatorname{cl} A \setminus A$ is not closed in X. Then there exists a point $x \in \operatorname{cl}(\operatorname{cl} A \setminus A) \setminus (\operatorname{cl} A \setminus A)$. It follows that $x \in A$. Let U be an arbitrary neighborhood of x in X. We will prove that $A \cap U$ is not closed in U. Since $x \in \operatorname{cl}(\operatorname{cl} A \setminus A)$, we can find a point $y \in (\operatorname{cl} A \setminus A) \cap U$. Let V be an open neighborhood of y in X. Then also $V \cap U$ is a neighborhood of y in X. Since $y \in \operatorname{cl}(\operatorname{cl} A \setminus A) \subset \operatorname{cl} A$, we see that $V \cap U \cap A \neq \varnothing$. Since all open neighborhoods of y in U are of the form $V \cap U$ where V is open in X, we see that $y \in \operatorname{cl}_U (U \cap A)$. However, $y \notin A$, hence, $y \notin U \cap A$. It follows that $U \cap A$ is not closed in U. This holds for every neighborhood U of x in X. Therefore, A is not locally closed. □

The topology $\mathcal{T}$ is T_2 or *Hausdorff* if for any two different points $x, y \in X$ there exist disjoint sets $U, V \in \mathcal{T}$ such that $x \in U$ and $y \in V$. It is T_0 or *Kolmogorov* if for any two different points $x, y \in X$ there exists a $U \in \mathcal{T}$ such that $U \cap \{x, y\}$ is a singleton.

A *topological space* is a pair $(X, \mathcal{T})$ where $\mathcal{T}$ is a topology on X. We often refer to X as a topological space assuming that the topology $\mathcal{T}$ on X is clear from context. Given $Y \subset X$, the family $\mathcal{T}_Y := \{\, U \cap Y \mid U \in \mathcal{T}\,\}$ is a topology on Y called the induced topology. The topological space $(Y, \mathcal{T}_Y)$ is called a subspace of $(X, \mathcal{T})$. For $A \subset Y$ we write $\operatorname{cl}_Y A := \operatorname{cl}_{\mathcal{T}_Y} A$.

3.6. **Finite topological spaces.** We say that a topological space X is a *finite topological space* if X is finite. Finite topological spaces differ from general topological spaces because the only Hausdorff topology on a finite topological space X is the discrete topology consisting of all subsets of X. In consequence, unlike in Hausdorff topological spaces, the closure of a singleton need not be a singleton. Moreover, for any $A \subset X$ we have

$$\operatorname{cl} A = \bigcup_{x \in A} \operatorname{cl} x \tag{1}$$

where we abbreviate $\operatorname{cl}\{x\}$ to the more convenient notation $\operatorname{cl} x$. A remarkable feature of finite topological spaces is the following theorem.

Theorem 3.5 (P. Alexandrov [1])**.** *For a preorder $\leq$ on a finite set X, there exists a topology $\mathcal{T}_{\leq}$ on X whose open sets are precisely the upper sets with respect to $\leq$. Conversely, if for a given topology $\mathcal{T}$ on a finite set X we define*

$$x \leq_{\mathcal{T}} y \qquad \Longleftrightarrow \qquad x \in \operatorname{cl}_{\mathcal{T}} y\ ,$$

then $\leq_{\mathcal{T}}$ is a preorder. The correspondences $\mathcal{T} \mapsto \leq_{\mathcal{T}}$ and $\leq \mapsto \mathcal{T}_{\leq}$ are mutually inverse. Under these correspondences, continuous maps are transformed into order-preserving maps, and vice versa. Moreover, the topology $\mathcal{T}$ is T_0 if and only if the preorder $\leq_{\mathcal{T}}$ is a partial order.

Locally closed sets have a special characterization in finite topological spaces.

Proposition 3.6. *Assume X is a finite topological space and $A \subset X$. Then the set A is locally closed if and only if for all $x, y, z \in X$ one has*

$$x, z \in A, \quad x \in \operatorname{cl} y, \quad y \in \operatorname{cl} z \quad \Rightarrow \quad y \in A. \tag{2}$$

Proof: Assume $A \subset X$ is locally closed, $x, z \in A$, $x \in \operatorname{cl} y$, $y \in \operatorname{cl} z$, as well as $y \notin A$. Then $y \in \operatorname{mo} A$. Since $x \in \operatorname{cl} y$, we get from Proposition 3.4 that $x \in \operatorname{mo} A$. In particular, one obtains $x \notin A$, which is a contradiction proving (2). To prove the opposite implication, assume that a subset $A \subset X$ satisfies assumption (2). By Proposition 3.4 it suffices to prove that $\operatorname{mo} A$ is closed. For this, take an $x \in \operatorname{cl} \operatorname{mo} A$ and assume $x \notin \operatorname{mo} A$. Since $\operatorname{cl} \operatorname{mo} A \subset \operatorname{cl} A$, we get $x \in \operatorname{cl} A$. Since $x \notin \operatorname{mo} A$, one has $x \in A$. Thus, we get from (1) that $x \in \operatorname{cl} y$ for a $y \in \operatorname{mo} A$. In particular, $y \in \operatorname{cl} A$ and by (1) $y \in \operatorname{cl} z$ for a $z \in A$. Hence, it follows from (2) that $y \in A$. But, $y \in \operatorname{mo} A$ implies that $y \notin A$, a contradiction proving that $\operatorname{mo} A$ is closed. $\square$

3.7. **Modules.** For the terminology used in the paper concerning modules we refer the reader to [14]. Here, we briefly recall that a *module* over a ring R is a triple $(M, +, \cdot)$ such that $(M, +)$ is an abelian group and $\cdot : R \times M \to M$ is a scalar multiplication which is distributive with respect to the additions in M and R, and which satisfies both $a \cdot (b \cdot x) = (ab) \cdot x$ and $1_R \cdot x = x$ for $x \in M$ and $a, b \in R$. We shorten the notation $a \cdot x$ to ax. We often say that M is a module assuming that the operations $+$ and $\cdot$ are clear from context. Recall that a *submodule* of M is a subset $N \subset M$ such that N with the operations $+$ and $\cdot$ restricted to N is itself again a module. Finally, for submodules $N_1, N_2, \ldots, N_k$ of a given module M, their *algebraic sum*

$$N_1 + N_2 + \cdots N_k := \{\, x_1 + x_2 + \ldots + x_k \in M \mid x_i \in N_i \,\}$$

is easily seen to be a submodule of M. We call it a *direct sum* and denote it by the symbol $N_1 \oplus N_2 \oplus \ldots \oplus N_k$, if for each $x \in N_1 + N_2 + \ldots + N_k$ the representation $x = x_1 + x_2 + \ldots + x_k$ with $x_i \in N_i$ is unique. We note that for a module M the family of submodules of M, denoted by $\operatorname{Sub}(M)$, forms a lattice $(\operatorname{Sub}(M), 0, M, \cap, +)$.

A subset $Z \subset M$ *generates* M if for every element $x \in M \setminus \{0\}$ there exist elements $x_1, x_2, \ldots, x_n \in Z$ and $a_1, a_2, \ldots, a_n \in R$ such that $x = \sum_{i=1}^n a_i x_i$. A subset $Z \subset M$ is *linearly independent* if for any choice of $x_1, x_2, \ldots, x_n \in Z$ and $a_1, a_2, \ldots, a_n \in R$ the equality $\sum_{i=1}^n a_i x_i = 0$ implies $a_1 = a_2 = \ldots = 0$. A linearly independent subset $B \subset M$ which generates M is called a *basis* of M. A module may not have a basis. If it does have a basis, the module is called *free*. We note that $\varnothing$ is the unique basis of the trivial module $\{0\}$.

Assume that M is a free module and that $B \subset M$ is a fixed basis. Then we have the associated bilinear map $\langle \cdot, \cdot \rangle_B : M \times M \to R$, called *scalar product*, and defined on basis elements $b, b' \in B$ by $\langle b, b' \rangle_B := 0$ for $b \neq b'$, as well as $\langle b, b' \rangle_B := 1$ for $b = b'$. For an element $x \in M$ we define its *support* with respect to B as $|x|_B := \{\, b \in B \mid \langle x, b \rangle_B \neq 0 \,\}$. One easily verifies that for all $x, y \in M$

$$|x + y|_B \subset |x|_B \cup |y|_B. \tag{3}$$

Assume that X is a finite set. Then the collection $R\langle X \rangle := \{\, f : X \to R \,\}$ of all functions forms a module with respect to pointwise addition and scalar multiplication. We note that $R\langle \varnothing \rangle$ is the zero module. For every $x \in X$

we have a function $\bar{x} : M \to R$ which sends x to 1 and any other element of X to 0. One easily verifies that $\bar{X} := \{\, \bar{x} \mid x \in X \,\}$ is a basis of $R\langle X\rangle$. Therefore, $R\langle X\rangle$ is a free module, and we call $R\langle X\rangle$ the *free module spanned by* X. In the sequel we identify $\bar{x}$ with x.

Let M and M' be two modules over R. A map $h : M \to M'$ is called a *module homomorphism* if we have $h(a_1x_1 + a_2x_2) = a_1h(x_1) + a_2h(x_2)$ for arbitrary $x_1, x_2 \in M$ and $a_1, a_2 \in R$. The *kernel* of h, denoted by $\ker h$, is the submodule $\{\, x \in M \mid h(x) = 0 \,\}$, while the *image* of h, denoted by $\operatorname{im} h$, is the submodule $\{\, y \in M' \mid \exists_{x\in M} h(x) = y \,\}$.

4. Poset graded and filtered modules

4.1. **Gradation of modules.** Assume R is a fixed ring and M is a module over R. Let K be an arbitrary set. Then a K*-gradation* of M is a collection of submodules $\{M_k\}_{k\in K}$ of M indexed by K, and such that $M = \bigoplus_{k\in K} M_k$ is the direct sum of the submodules M_k. This definition requires that $K \neq \varnothing$, but it is convenient to assume that the direct sum over the empty set is always the zero module. Thus, the only module admitting the $\varnothing$-gradation is the zero module. By a K*-graded module* over R we mean a module M over R together with a fixed, implicitly given, K-gradation. Note that the zero module admits not only the $\varnothing$-gradation, but also a unique K-gradation for each non-empty set K. Thus, for each set K the zero module is a K-graded module. We denote it by 0_K.

If B is a basis of a free module M and $\mathcal{A}$ is a partition of B, then M is $\mathcal{A}$-graded with the gradation

$$M = \bigoplus_{A\in\mathcal{A}} R\cdot A,$$

where the expression on the right-hand side is defined as

$$R\cdot A = \left\{ \sum_{i=1}^{n} r_i a_i \;\mid\; r_i \in R,\ a_i \in A,\ i = 1,\dots,n,\ n \in \mathbb{N} \right\}.$$

Note that in the case that A is finite, one can identify $R\cdot A$ with $R\langle A\rangle$. As a special case, consider the situation when the partition of B consists only of singletons. Then the gradation becomes

$$M = \bigoplus_{b\in B} R\cdot b.$$

We refer to this gradation as the B*-basis gradation* of M.

We say that a submodule M' of M is a K*-graded submodule* if M' is also K-graded with the decomposition

$$M' = \bigoplus_{k\in K} M'_k$$

and such that $M'_k \subset M_k$. For $I \subset K$ we have an *I-graded submodule*

$$M_I := \bigoplus_{i \in I} M_i.$$

The homomorphisms

$$\iota_I : M_I \ni x \mapsto x \in M$$

and

$$\pi_I : M = \bigoplus_{k \in K} M_k \ni \sum_{k \in K} x_k \mapsto \sum_{i \in I} x_i \in M_I$$

are called the associated *canonical inclusion* and *canonical projection*. In the case $I = \{ i \}$ we abbreviate the above notation to ι_i and π_i.

Assume K' is another set and M' is a K'-graded module. For a module homomorphism $f : M \to M'$ and subsets $J \subset K$ and $I \subset K'$ we have an induced homomorphism $f_{IJ} : M_J \to M'_I$ defined as the composition

$$f_{IJ} := \pi_I \circ f \circ \iota_J,$$

where $\pi_I : M' \to M'_I$ and $\iota_J : M_J \to M$. Again, if $I = \{i\}$ and $J = \{j\}$ we abbreviate the notation to f_{ij}. One easily verifies the following proposition.

Proposition and Definition 4.1. *Assume K and K' are finite. Then*

$$f = \sum_{i \in K'} \iota_i \sum_{j \in K} f_{ij} \pi_j. \tag{4}$$

In particular, f is uniquely determined by the matrix $[f_{ij}]_{i \in K', j \in K}$ of homomorphisms $f_{ij} : M_j \to M'_i$. We refer to this matrix as the (K, K')-matrix *of the homomorphism f.* □

Notice that in the case when the K- and K'-gradations are basis gradations given by bases $B = \{b_1, b_2, \ldots, b_n\}$ in M and $B' = \{b'_1, b'_2, \ldots, b'_m\}$ in M', respectively, then the homomorphisms f_{ij} take the form

$$f_{ij} \ : \ t \cdot b_j \mapsto t \langle f(b_j), b_i \rangle \cdot b_i,$$

which means that the (K, K')-matrix of f may be identified in this case with the matrix of coefficients $\langle f(b_j), b_i \rangle$.

As in the case of classical matrices one easily verifies the following proposition.

Proposition 4.2. *Consider modules M, M', and M'' which are graded by K, K', and K'', respectively. Let $f : M \to M'$ and $f' : M' \to M''$ be module homomorphisms. Then the (K, K'')-matrix of f consists of the homomorphisms*

$$(f'f)_{ik} = \sum_{j \in K'} f'_{ij} f_{jk} \tag{5}$$

for $i \in K''$ and $k \in K$. □

4.2. **Graded and filtered homomorphisms.** Consider a K-graded module M and a K'-graded module M'. Let $\alpha : K' \nrightarrow K$ be a partial map. We say that a homomorphism $h : M \to M'$ is *α-graded* if for every pair of indices $j \in K$ and $i \in K'$ we have

$$h_{ij} \neq 0 \quad \Rightarrow \quad i \in \alpha^{-1}(j),$$

which is equivalent to the implication

$$h_{ij} \neq 0 \quad \Rightarrow \quad i \in \operatorname{dom}\alpha \quad \text{and} \quad \alpha(i) = j. \tag{6}$$

If in addition the sets K and K' are partially ordered sets and α is order preserving, then we say that a homomorphism $h : M \to M'$ is *α-filtered* if for every $j \in K$ and $i \in K'$

$$h_{ij} \neq 0 \quad \Rightarrow \quad i \in \alpha^{-1}(j^{\leq})^{\leq} \tag{7}$$

which is equivalent to

$$h_{ij} \neq 0 \quad \Rightarrow \quad \exists_{i' \in K' \cap \operatorname{dom}\alpha}\, i \leq i' \quad \text{and} \quad \alpha(i') \leq j.$$

These two definitions are illustrated in Figures 2 and 3. The left image in Figure 2 shows two partially ordered sets together with a partial map α between them. In the right image of the same figure arrows indicate which of the homomorphisms h_{ij} can be nontrivial if h is α-graded. In contrast, Figure 3 depicts all possible nontrivial homomorphisms h_{ij} if h is α-filtered. The more involved implication (7) is illustrated in the top left image, which indicates the sets $i^{\leq}$ and $j^{\leq}$ in light blue. The remaining three panels in the figure depict all possible nontrivial homomorphisms h_{ij}. Clearly, α-filtered homomorphisms are far less restricted than α-graded ones. In fact, an α-filtered homomorphism can have a nontrivial h_{ij} even if $i \notin \operatorname{dom}\alpha$ or if $j \notin \operatorname{im}\alpha$.

Conditions (6) and (7) for α-graded and α-filtered homomorphisms, respectively, may be simplified if $\operatorname{dom}\alpha = K'$, i.e., if α is actually a function. This is the subject of the following straightforward proposition.

Proposition 4.3. *Assume* $\operatorname{dom}\alpha = K'$. *Then $h : M \to M'$ is α-graded if and only if for every $i \in K'$ and $j \in K$*

$$h_{ij} \neq 0 \quad \Rightarrow \quad \alpha(i) = j, \tag{8}$$

and h is α-filtered if and only if

$$h_{ij} \neq 0 \quad \Rightarrow \quad \alpha(i) \leq j \tag{9}$$

for every $i \in K'$ and $j \in K$. □

Obviously, if α is order preserving and h is an α-graded homomorphism then h is an α-filtered homomorphism. However, the converse is not true in general. This can readily be seen from Figures 2 and 3.

We say that a homomorphism $h : M \to M'$ is *filtered* (respectively *graded*) if $K = K'$ and h is id_K-filtered (respectively id_K-graded). As an immediate consequence of Proposition 4.3 we get the following corollary.

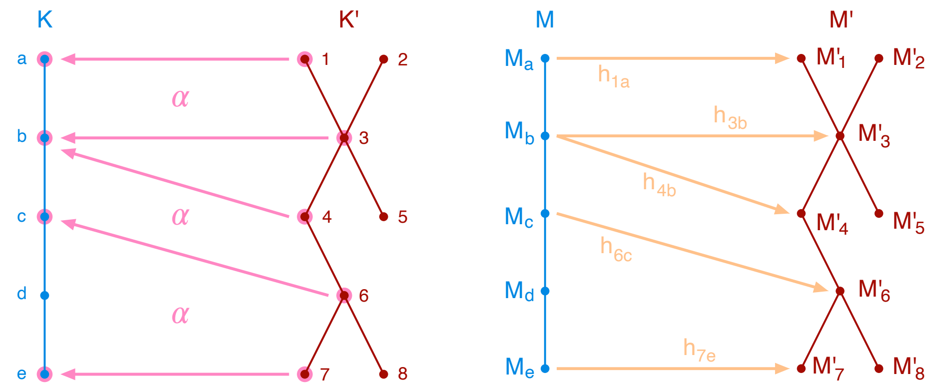

FIGURE 2. Sample α-graded morphism. The left image shows two posets K and K', together with an order preserving partial map $\alpha : K' \nrightarrow K$. The panel on the right indicates the potential nonzero module homomorphisms h_{ij} if h is α-graded.

Corollary 4.4. *An endomorphism $h : M \to M$ on a module M is graded if and only if for every $i, j \in K$ we have*

$$h_{ij} \neq 0 \quad \Rightarrow \quad i = j,$$

and h is filtered if and only if one has

$$h_{ij} \neq 0 \quad \Rightarrow \quad i \leq j$$

for every $i, j \in K$. □

Consider now the special case when $K = K' = \mathbb{Z}$ and $\alpha : \mathbb{Z} \ni i \mapsto i-k \in \mathbb{Z}$ is the left-shift by k on the integers. If in this situation h is α-graded, we say that it is a *$\mathbb{Z}$-graded homomorphism* of degree k. When k is clear from the context, we shorten the notation $h_{j+k,j}$ to h_j.

Proposition 4.5. *Assume that P and P' are posets, $\alpha : P' \nrightarrow P$ is order preserving, M is a P-graded module, and M' is a P'-graded module. A module homomorphism $h : M \to M'$ is an α-filtered homomorphism if and only if*

$$h(M_L) \subset M'_{\alpha^{-1}(L)^{\leq}} \tag{10}$$

for every $L \in \mathrm{Down}(P)$.

Proof: Assume that $h : M \to M'$ is a module homomorphism such that (10) is satisfied for every $L \in \mathrm{Down}(P)$. Moreover, assume that $h_{pq} \neq 0$ for an element $q \in P$ and a $p \in P'$. Let $x \in M_q$ be such that $h_{pq}(x) \neq 0$. If we consider $L := q^{\leq} \in \mathrm{Down}(P)$, then (10) implies $h(M_L) \subset M'_{\alpha^{-1}(L)^{\leq}}$. Since $x \in M_q \subset M_L$, we get $h(x) \in M'_{\alpha^{-1}(L)^{\leq}}$. Thus, $h_{p'q}(x) = \pi_{p'}(h(x)) = 0$ for $p' \notin \alpha^{-1}(L)^{\leq}$, which in turn implies that $p \in \alpha^{-1}(L)^{\leq}$, in view of $h_{pq}(x) =$

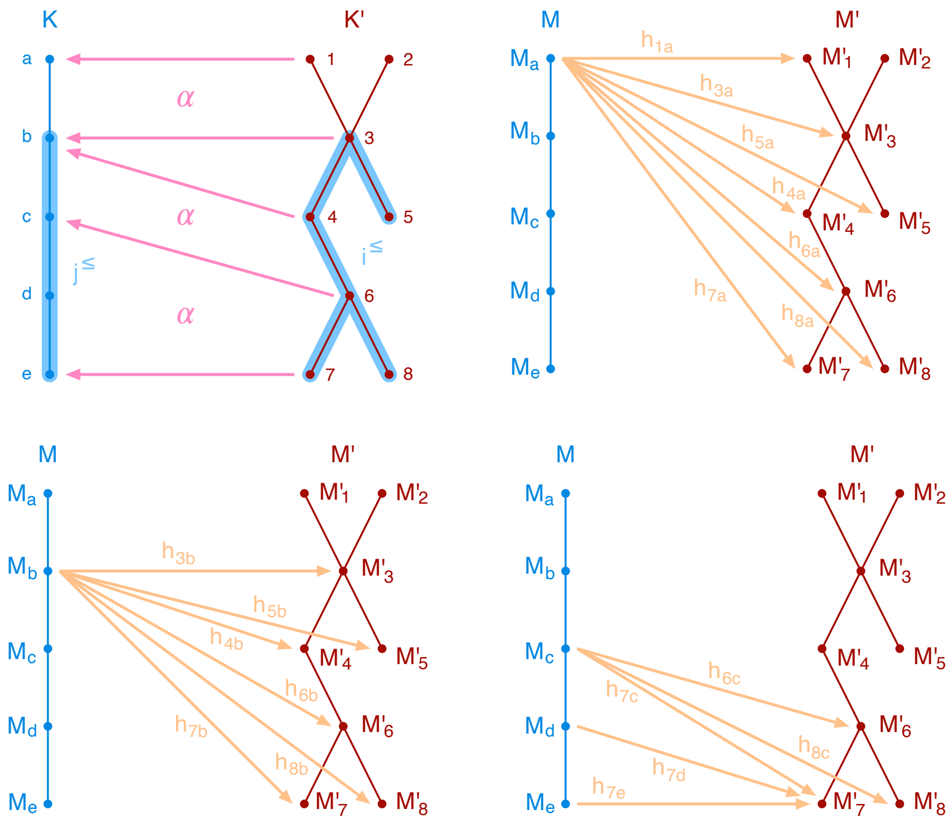

FIGURE 3. Sample α-filtered morphism. The top left image shows two posets K and K', together with an order preserving partial map $\alpha : K' \nrightarrow K$. The remaining three panels indicate all potentially nonzero module homomorphisms h_{ij} if h is α-filtered.

$\pi_p(h(x)) \neq 0$. In consequence, $p \leq \bar{p}$ for a $\bar{p}$ with $\alpha(\bar{p}) \in L = q^{\leq}$. Therefore, $p \in \alpha^{-1}(q^{\leq})^{\leq}$ which implies (7) and proves that h is α-filtered. To prove the opposite implication, assume that $h : M \to M'$ is a module homomorphism such that property (7) holds. Let $L \in \operatorname{Down}(P)$ and consider $x \in M_L$. Without loss of generality we may assume that $x \in M_q$ for a $q \in L$. Then an application of (7) yields

$$h(x) = \sum_{p \in P} h_{pq}(x) = \sum_{p \in \alpha^{-1}(q^{\leq})^{\leq}} h_{pq}(x),$$

which means that $h(x) \in M'_{\alpha^{-1}(L)^{\leq}}$, since we have $\alpha^{-1}(q^{\leq})^{\leq} \subset \alpha^{-1}(L)^{\leq}$. This establishes the inclusion in (10). $\square$

In the special case when $P = P'$ and $\alpha = \mathrm{id}_P$ we have the following corollary of Proposition 4.5.

Corollary 4.6. *Assume that P is a poset and that M is a P-graded module. Then $h : M \to M$ is a filtered homomorphism if and only if*

$$h(M_L) \subset M_L \tag{11}$$

for every $L \in \mathrm{Down}(P)$.

Proof: By applying our assumptions $P = P'$ and $\alpha = \mathrm{id}_P$ to the setting of Proposition 4.5, one obtains that $\alpha^{-1}(L)^{\leq} = L^{\leq} = L$. Therefore, the condition in (10) reduces to the one in (11). □

Definition 4.7. We say that two P-gradations $(M_p)_{p\in P}$ and $(M'_p)_{p\in P}$ of the same module M are *filtered equivalent*, if for every $L \in \mathrm{Down}(P)$ we have $M_L = M'_L$.

As an immediate consequence of Corollary 4.6 one obtains the following proposition.

Proposition 4.8. *Let P be a poset. Assume that M and M' are P-graded modules and that $h : M \to M'$ is a filtered homomorphism. Then h is a filtered homomorphism with respect to any filtered equivalent gradation of the modules M and M'.* □

4.3. **The category of graded and filtered moduli.** We define the two categories GMod of graded moduli and FMod of filtered moduli as follows. The objects of GMod (respectively FMod) are pairs (P, M) where P is an object of DSet (respectively DPSet) and M is a P-graded module; see also Sections 3.1 and 3.3. The morphisms in GMod (respectively FMod) are pairs $(\alpha, h) : (P, M) \to (P', M')$ such that $\alpha : P' \nrightarrow P$ is a morphism in DSet (respectively DPSet) and $h : M \to M'$ is an α-graded (respectively α-filtered) module homomorphism. In the following, we will briefly refer to morphisms in GMod (respectively FMod) as graded (respectively filtered) morphisms $(\alpha, h) : (P, M) \to (P', M')$. In the special case when $(P, M) = (P', M')$ and $\alpha = \mathrm{id}_P$ we simplify the terminology be referring to h as a filtered (respectively graded) morphism meaning that h is id_P-filtered (respectively id_P-graded).

One can easily verify that $\mathrm{id}_{(P,M)} := (\mathrm{id}_P, \mathrm{id}_M)$ is the identity morphism on (P, M) in both GMod and FMod. Also, if $(\alpha, h) : (P, M) \to (P', M')$ and $(\alpha', h') : (P', M') \to (P'', M'')$ are two graded or filtered morphisms, then we define their composition

$$(\alpha'', h'') := (\alpha', h') \circ (\alpha, h) : (P, M) \to (P'', M'')$$

by $\alpha'' := \alpha\alpha'$ and $h'' := h'h$. This leads to the following fundamental results.

Proposition 4.9. GMod *is a well-defined category.*

Proof: The only nontrivial part of the proof is the verification that the composition of two graded morphisms is again a graded morphism. For this, let $(\alpha'', h'') := (\alpha', h') \circ (\alpha, h)$ be given as above. Obviously $h'h$ is a module homomorphism. We will show that $h''_{pr} \neq 0$ for $p \in P''$ and $r \in P$ implies

$(\alpha\alpha')(p) = r$. Hence, assume that $h''_{pr} \neq 0$ for $p \in P''$ and $r \in P$. It follows from Proposition 4.2 that there exists a $q \in P'$ such that both $h'_{pq} \neq 0$ and $h_{qr} \neq 0$ are satisfied. Therefore, one has to have $p \in \operatorname{dom} \alpha'$ and $\alpha'(p) = q$ as well as $q \in \operatorname{dom} \alpha$ and $\alpha(q) = r$. It follows that $p \in \operatorname{dom}(\alpha\alpha')$ and $(\alpha\alpha')(p) = r$. This proves that (α'', h'') is a graded morphism. □

A given pair of poset graded morphisms $f : (P, M) \to (P, M)$ and $f' : (P', M') \to (P', M')$ are called *graded-conjugate*, if there exists a poset graded isomorphism $(\alpha, h) : (P, M) \to (P', M')$ such that

$$(\alpha, h) \circ f = f' \circ (\alpha, h).$$

If f and f' are graded-conjugate, then we say that their respective (P, P) and (P', P') matrices are *graded-similar*. It is easily seen that the graded similarity of matrices is an equivalence relation.

Proposition 4.10. FMOD *is a well-defined category.*

Proof: The only nontrivial property to verify is that the composition of filtered morphisms is again a filtered morphism. For this, consider a composition $(\alpha'', h'') := (\alpha', h') \circ (\alpha, h)$ of filtered morphisms as above. Obviously, the map $\alpha\alpha'$ is order preserving and $h'h$ is a module homomorphism. Thus, we only need to verify that that $h''_{pr} \neq 0$ for $p \in P''$ and $r \in P$ implies the inclusion $p \in (\alpha\alpha')^{-1}(r^{\leq})^{\leq}$. Hence, assume that $h''_{pr} \neq 0$ for $p \in P''$ and $r \in P$. It follows from Proposition 4.2 that there exists a $q \in P'$ such that $h'_{pq} \neq 0$ and $h_{qr} \neq 0$. Therefore, there exist $\bar{p} \geq p$ and $\bar{q} \geq q$ such that $\alpha'(\bar{p}) \leq q$ and $\alpha(\bar{q}) \leq r$. Since α is order preserving, it follows that $(\alpha\alpha')(\bar{p}) \leq \alpha(\bar{q}) \leq r$. Hence, $\bar{p} \in (\alpha\alpha')^{-1}(r^{\leq})$ and $p \in (\alpha\alpha')^{-1}(r^{\leq})^{\leq}$. This proves that (α'', h'') is a filtered morphism. □

Proposition 4.11. *Assume that the two maps* $(\alpha, h) : (P, M) \to (P', M')$ *and* $(\alpha', h') : (P', M') \to (P, M)$ *are morphisms in* FMOD *which satisfy the identities* $\operatorname{dom} \alpha = P'$ *and* $\operatorname{dom} \alpha' = P''$*. Suppose further that* α *is injective. If* $p \in P$ *and* $p'' \in P''$ *are such that* $\alpha\alpha'(p'') = p$*, then*

$$(h'h)_{p''p} = h'_{p''\alpha'(p'')} h_{\alpha'(p'')p} \quad \textit{for } p \in P. \tag{12}$$

Proof: We get from Proposition 4.2 that

$$(h'h)_{p''p} = \sum_{p' \in P'} h'_{p''p'} h_{p'p}. \tag{13}$$

It follows from (9) that the index p' of a non-zero term on the right-hand side of (13) must satisfy $\alpha'(p'') \leq p'$ and $\alpha(p') \leq p$. Hence,

$$p = \alpha\alpha'(p'') \leq \alpha(p') \leq p.$$

It follows that $\alpha(p') = p$ and, since α is injective, we get $p' = \alpha'(p'')$. This proves (12). □

As an immediate consequence of Proposition 4.11 we get the following corollary.

Corollary 4.12. *Assume that the two maps $(\alpha, h) : (P, M) \to (P', M')$ and $(\alpha', h') : (P', M') \to (P, M)$ are morphisms in* FMod *such that α and α' are mutually inverse bijections. Then*

$$(h'h)_{pp} = h'_{p\alpha'(p)} h_{\alpha'(p)p} \quad \textit{for } p \in P.$$

Lemma 4.13. *A filtered homomorphism $(\alpha, h) : (P, M) \to (P', M')$ is an isomorphism in* FMod *if and only if $\alpha : P' \to P$ is an order isomorphism and $h_{p'\alpha(p')} : M_{\alpha(p')} \to M'_{p'}$ is a module isomorphism for every $p' \in P'$.*

Proof: First assume that (α, h) is an isomorphism in the category FMod. Let $(\alpha', h') : (P', M') \to (P, M)$ be the inverse of (α, h). Then $\alpha\alpha' = \mathrm{id}_P$ and $\alpha'\alpha = \mathrm{id}_{P'}$. Since α and α' are order preserving, we see that $\alpha : P' \to P$ is an order isomorphism. It follows from Corollary 4.12 that

$$\mathrm{id}_{M_p} = (\mathrm{id}_M)_{pp} = (h'h)_{pp} = h'_{p\alpha'(p)} h_{\alpha'(p)p} \text{ for } p \in P \tag{14}$$

and

$$\mathrm{id}_{M_{p'}} = (\mathrm{id}_M)_{p'p'} = (hh')_{p'p'} = h_{p'\alpha(p')} h'_{\alpha(p')p'} \text{ for } p' \in P'. \tag{15}$$

But, α and α' are mutually inverse bijections. Hence, substituting $p' := \alpha'(p)$ in (14) we get $p = \alpha(p')$ and

$$\mathrm{id}_{M_{\alpha(p')}} = h'_{\alpha(p')p'} h_{p'\alpha(p')} \text{ for } p' \in P'.$$

It follows that $h'_{\alpha(p')p'}$ is the inverse of $h_{p'\alpha(p')}$. Therefore, $h_{p'\alpha(p')}$ is a module isomorphism for every $p' \in P'$.

To see the opposite implication, assume that $(\alpha, h) : (P, M) \to (P', M')$ is a filtered homomorphism such that $\alpha : P' \to P$ is an order isomorphism and $h_{p'\alpha(p')} : M_{\alpha(p')} \to M'_{p'}$ is a module isomorphism for all $p' \in P'$. Since P and P' as objects of DPSet are finite sets, we may proceed by induction on their cardinality $n := \operatorname{card} P = \operatorname{card} P'$. If $n = 1$, then $P = \{p\}$, $P' = \{p'\}$ and $h_{p'\alpha(p')} = h_{p'p} = h$ is a module isomorphism. Clearly, its inverse is a filtered homomorphism. Hence, h is an isomorphism in FMod. Thus, we now assume $n > 1$. Let $\bar{p}'$ be a maximal element in P'. Set $\bar{p} := \alpha(\bar{p}')$. Since α' is an order isomorphism, we see that $\bar{p}$ is a maximal element in P. Let $\bar{P}' := P' \setminus \{\bar{p}'\}$, $\bar{P} := P \setminus \{\bar{p}\}$, $\bar{M}' := M_{\bar{P}'}$, $\bar{M} := M_{\bar{P}}$. Let $Q := \{\bar{P}, \bar{p}\}$ and $Q' := \{\bar{P}', \bar{p}'\}$ be linearly ordered respectively by $\bar{P} < \bar{p}$ and $\bar{P}' < \bar{p}'$. Then (Q, M) and (Q', M') are graded modules. Define $\bar{\alpha} : Q' \to Q$ through the identities $\bar{\alpha}(\bar{P}') := \bar{P}$ and $\bar{\alpha}(\bar{p}') := \bar{p}$. We will prove that

$$h_{\bar{p}'p} = 0 \text{ for } p \in \bar{P}. \tag{16}$$

Arguing by contradiction, assume that $h_{\bar{p}'p} \neq 0$ for a $p \in \bar{P}$. Since h is an α-filtered homomorphism, we get from (7) that $h_{\bar{p}'p} \neq 0$ implies $\bar{p}' \leq \bar{\bar{p}}'$ for some $\bar{\bar{p}}' \in \operatorname{dom} \alpha$ such that $\alpha(\bar{\bar{p}}') \leq p$. Since $\bar{p}'$ is maximal in P', we obtain both $\bar{p}' = \bar{\bar{p}}'$ and $\bar{p} = \alpha(\bar{p}') \leq p$. Since $\bar{p}$ is maximal in P, this further implies the equality $p = \bar{p}$, a contradiction proving (16). Hence, the identity in (4) yields $h_{\bar{p}'\bar{P}} = \sum_{p\in\bar{P}} h_{\bar{p}'p} \circ \pi_p = 0$. Therefore, the (Q, Q')-matrix of h is

$$\begin{bmatrix} h_{\bar{P}'\bar{P}} & h_{\bar{P}'\bar{p}} \\ 0 & h_{\bar{p}'\bar{p}} \end{bmatrix}.$$

By induction assumption $(\alpha_{|\bar{P}'}, h_{\bar{P}'\bar{P}}) : M_{\bar{P}} \to M_{\bar{P}'}$ is an isomorphism in FMod. Set $\alpha' := \alpha^{-1}$ and let $h' : M' \to M$ be the homomorphism given by the (Q', Q)-matrix

$$\begin{bmatrix} h^{-1}_{\bar{P}'\bar{P}} & -h^{-1}_{\bar{P}'\bar{P}} h_{\bar{P}'\bar{p}} h^{-1}_{\bar{p}'\bar{p}} \\ 0 & h^{-1}_{\bar{p}'\bar{p}} \end{bmatrix}.$$

One easily verifies that h' is α'-filtered and a straightforward computation shows that $(\alpha', h') \circ (\alpha, h) = \mathrm{id}_{(P,M)}$ and $(\alpha, h) \circ (\alpha', h') = \mathrm{id}_{(P',M')}$. Hence, the morphism (α, h) is indeed an isomorphism in FMod. $\square$

Corollary 4.14. *Assume $(\alpha, h) : (P, M) \to (P', M')$ is both a homomorphism in* GMod *and an isomorphism in* FMod*. Then it is automatically an isomorphism in* GMod*.*

Proof: Due to Lemma 4.13 the map $\alpha : P' \to P$ is an order isomorphism and $h_{p'\alpha(p')} : M_{\alpha(p')} \to M'_{p'}$ is a module isomorphism. Let $g : M' \to M$ be the α^{-1}-graded homomorphism with (P', P)-matrix given by

$$g_{pp'} := \begin{cases} h^{-1}_{p'\alpha(p')} & \text{if } p = \alpha(p'), \\ 0 & \text{otherwise.} \end{cases}$$

It is straightforward to verify that (α^{-1}, g) is the inverse of (α, h) in GMod. Hence, (α, h) is an isomorphism in GMod. $\square$

5. Poset filtered chain complexes

5.1. **Chain complexes.** Recall that a *chain complex* is a pair (C, d) consisting of a $\mathbb{Z}$-graded R-module C and a $\mathbb{Z}$-graded homomorphism $d : C \to C$ of degree -1, satisfying $d^2 = 0$. The homomorphism d is called the *boundary homomorphism* of the chain complex (C, d). Let (C', d') be another chain complex. A *chain map* $\varphi : (C, d) \to (C', d')$ is a module homomorphism $\varphi : C \to C'$ of degree zero, satisfying $\varphi d = d'\varphi$. The following proposition is straightforward.

Proposition 5.1. *If $\varphi : (C, d) \to (C', d')$ is a chain map and φ is an isomorphism of $\mathbb{Z}$-graded modules then φ^{-1} is also a chain map.* $\square$

We denote by Cc the category whose object are chain complexes and whose morphisms are chain maps. One easily verifies that this is indeed a category.

A subset $C' \subset C$ is a *chain subcomplex* of C if C' is a $\mathbb{Z}$-graded submodule of C such that $d(C') \subset C'$. Recall that given a subcomplex $C' \subset C$ we have a well-defined *quotient complex* $(C/C', d')$ where $d' : C/C' \to C/C$ is the boundary homomorphism induced by d.

5.2. **Homotopy category of chain complexes.** A pair of chain maps $\varphi, \varphi' : (C, d) \to (C', d')$ are called *chain homotopic* if there exists a *chain homotopy* joining φ and φ', i.e., a module homomorphism $S : C \to C'$ of degree $+1$ such that $\varphi' - \varphi = d'S + Sd$. The existence of a chain homotopy between two chain maps is easily seen to be an equivalence relation in the set of chain maps $\mathrm{Cc}((C, d), (C', d'))$. Given a chain map $\varphi \in \mathrm{Cc}((C, d), (C', d'))$ we denote by $[\varphi]$ the equivalence class of φ with respect to this equivalence relation. We define the *homotopy category* ChCc of chain complexes by taking chain complexes as objects, equivalence classes of morphisms in Cc as morphisms in ChCc, and the formula

$$[\psi] \circ [\varphi] := [\psi\varphi] \tag{17}$$

for $\psi \in \mathrm{Cc}((C', d'), (C'', d''))$ as the definition of composition of morphisms in ChCc. Note that then the equivalence classes of identities in Cc are the identities in ChCc.

Proposition 5.2. *The category* ChCc *is well-defined.*

Proof: The only non-obvious part of the argument is the verification that the composition given by (17) is well-defined. Thus, we need to prove that if $\varphi, \varphi' : (C, d) \to (C', d')$ and $\psi, \psi' : (C', d') \to (C'', d'')$ are chain homotopic, then $\psi\varphi$ and $\psi'\varphi'$ are chain homotopic. Let $S : C \to C'$ and $S' : C' \to C''$ be chain homotopies between φ, φ' and ψ, ψ', respectively. Moreover, consider the map $S'' := \psi' S + S'\varphi$. Then S'' is clearly a degree $+1$ homomorphism and we have

$$\begin{aligned} \psi'\varphi' - \psi\varphi &= \psi'(\varphi' - \varphi) + (\psi' - \psi)\varphi = \psi'(d'S + Sd) + (d''S' + S'd')\varphi \\ &= \psi'd'S + \psi'Sd + d''S'\varphi + S'd'\varphi \\ &= d''\psi'S + \psi'Sd + d''S'\varphi + S'\varphi d = d''S'' + S''d, \end{aligned}$$

which proves that $\psi\varphi$ and $\psi'\varphi'$ are indeed chain homotopic. □

We have a covariant functor $\mathrm{Ch} : \mathrm{Cc} \to \mathrm{ChCc}$ which fixes objects and sends a chain map to its chain homotopy equivalence class. Moreover, we say that two chain complexes (C, d) and (C', d') are *chain homotopic* if they are isomorphic in ChCc.

Note that $0_{\mathbb{Z}}$, the $\mathbb{Z}$-graded zero module, together with zero homomorphism as the boundary map, is a chain complex. We call it the *zero chain complex.* We say that a chain complex (C, d) is *homotopically essential* if it is not chain homotopic to the zero chain complex. Otherwise we say that the chain complex (C, d) is *homotopically trivial* or *homotopically inessential.* Finally, we call a chain complex (C, d) *boundaryless* if $d = 0$.

Proposition 5.3. *Assume (C, d) and (C', d') are two boundaryless chain complexes. Then the chain complexes (C, d) and (C', d') are chain homotopic if and only if C and C' are isomorphic as $\mathbb{Z}$-graded modules.*

Proof: Suppose that the chain complexes (C, d) and (C', d') are boundaryless. First assume that C and C' are isomorphic as $\mathbb{Z}$-graded modules.

Let $\varphi : C \to C'$ and $\varphi' : C' \to C$ be mutually inverse isomorphisms of $\mathbb{Z}$-graded modules. Since both boundary homomorphisms d and d' are zero, we have $\varphi d = 0 = d'\varphi$ and $\varphi' d' = 0 = d\varphi'$. Hence, φ and φ' are chain maps and from $\varphi'\varphi = \mathrm{id}_C$ and $\varphi\varphi' = \mathrm{id}_{C'}$ we get $[\varphi'][\varphi] = [\mathrm{id}_C]$ and $[\varphi][\varphi'] = [\mathrm{id}_{C'}]$. This shows that $[\varphi]$ and $[\varphi']$ are mutually inverse isomorphisms in ChCc.

To prove the opposite implication assume that (C, d) and (C', d') are chain homotopic. Choose $\varphi : (C, d) \to (C', d')$ and $\varphi' : (C', d') \to (C, d)$ such that $[\varphi] : (C, d) \to (C', d')$ and $[\varphi] : (C', d') \to (C, d)$ are mutually inverse isomorphisms in ChCc. Let $S : C \to C$ and $S' : C' \to C'$ be the chain homotopies between $\varphi'\varphi$ and id_C, and between $\varphi\varphi'$ and $\mathrm{id}_{C'}$, respectively. Then $\mathrm{id}_C - \varphi'\varphi = Sd + dS = 0$ and $\mathrm{id}_{C'} - \varphi\varphi' = S'd' + d'S' = 0$, which proves that φ and φ' are mutually inverse isomorphisms in Cc. □

Corollary 5.4. *The only chain complex which is both boundaryless and homotopically trivial is the zero chain complex.* □

Recall that the *homology module* of a chain complex (C, d) is the $\mathbb{Z}$-graded module $H(C) := (H_n(C, d))_{n\in\mathbb{Z}}$ where $H_n(C, d) := \ker d_n / \operatorname{im} d_{n+1}$. In the sequel, we will consider the homology module as a boundaryless chain complex, that is, as a chain complex with zero boundary homomorphism.

By a *homology decomposition* of a chain complex (C, d) we mean a direct sum decomposition $C = V \oplus H \oplus B$ such that V, H, B are $\mathbb{Z}$-graded submodules of C, $d_{|H} = 0$, $d(V) \subset B$ and $d_{|V} : V \to B$ is a module isomorphism. Note that then also $d_{|B} = 0$. By a *homology complex* of a chain complex (C, d) we mean a boundaryless chain complex which is chain homotopic to (C, d). Then one has the following proposition.

Proposition 5.5. *Assume that R is a field and that the pair (C, d) is a chain complex over R.*

(i) *There exists a homology decomposition of (C, d).*
(ii) *If $C = V \oplus H \oplus B$ is a homology decomposition of the chain complex (C, d), then $(H, 0)$ is chain homotopic to (C, d). Therefore, it is a homology complex of (C, d). Moreover, the modules H and $H(C)$ are isomorphic as $\mathbb{Z}$-graded modules, where $H(C)$ denotes the homology module of C.*

Proof: To prove statement (i), let $Z := \ker d$ and $B := \operatorname{im} d$. Since R is a field, we may choose $\mathbb{Z}$-graded submodules $V \subset C$ and $H \subset Z$ such that both $C = V \oplus Z$ and $Z = H \oplus B$ are satisfied, thus also $d_{H\oplus B} = 0$. Clearly, one has $d(V) \subset \operatorname{im} d = B$. We will prove that $d_{|V} : V \to B$ is an isomorphism. Indeed, if $dx = 0$ for an $x \in V$, then $x \in Z \cap V = \{0\}$, hence $d_{|V}$ is a monomorphism. To see that it is also onto, take a $y \in B$. Since $B = \operatorname{im} d$, we have $y = dx$ for an $x \in C$. But, $x = v + z$ for a $v \in V$ and a $z \in Z$. It follows that $dv = dv + dz = dx = y$, which establishes $d_{|V}$ as an epimorphism, and proves (i).

To prove (ii), assume that $C = V \oplus H \oplus B$ is a homology decomposition of (C, d). Let $\iota : H \to C$ denote inclusion, and let $\pi : C \to H$ be the projection map defined via $\pi(v+h+b) = h$ for $v+h+b \in V \oplus H \oplus B = C$. One can easily see that $\iota : (H,0) \to (C,d)$ and $\pi : (C,d) \to (H,0)$ are chain maps, and that $\pi\iota = \mathrm{id}_H$. We will show that $\iota\pi$ is chain homotopic to id_C. For this, define the degree $+1$ homomorphism $S : C \to C$ by $Sx = d_{|V}^{-1}b$, where $x = v+h+b \in C = V \oplus H \oplus B$. Then we have

$$\begin{aligned}(dS+Sd)x &= (dS+Sd)(v+h+b) = dd_{|V}^{-1}b + d_{|V}^{-1}dv = b+v \\ &= (b+v+h) - h = x - \iota\pi x = (\mathrm{id}_C - \iota\pi)x,\end{aligned}$$

which proves that S is a chain homotopy joining $\iota\pi$ and id_C. Finally, directly from the homology decomposition definition one obtains $\ker d = H \oplus B$ and $\operatorname{im} d = B$. Therefore, $H(C) = \ker d / \operatorname{im} d \cong H \oplus B/B \cong H$, where all the isomorphisms are $\mathbb{Z}$-graded. □

In the case of field coefficients the concepts of homology module and homology complex are essentially the same as the following theorem shows.

Theorem 5.6. *Assume that R is a field. Then the following hold.*

(i) *Every chain complex admits a homology complex.*
(ii) *Two chain complexes are chain homotopic if and only if the associated homology complexes are isomorphic as $\mathbb{Z}$-graded modules.*
(iii) *The homology module of a chain complex is its homology complex.*
(iv) *Two chain complexes are chain homotopic if and only if the associated homology modules are isomorphic as $\mathbb{Z}$-graded modules.*

Proof: Property (i) follows immediately from Proposition 5.5. In order to prove (ii), observe that two chain complexes are chain homotopic if and only if the associated homology complexes are chain homotopic. Therefore, property (ii) follows immediately from Proposition 5.3. To prove (iii), take a chain complex (C,d). By (i) we may consider a homology complex $(H,0)$ of (C,d). Since (C,d) and $(H,0)$ are chain homotopic, by a standard theorem of homology theory [9, §13] the homology modules of (C,d) and $(H,0)$ are isomorphic as $\mathbb{Z}$-graded modules. Hence, it follows from Proposition 5.5(ii) that the homology module of (C,d) is its homology complex. Finally, property (iv) follows immediately from (ii) and (iii). □

5.3. **Poset filtered chain complexes.** Let P be an arbitrary finite set and (C,d) a chain complex. We call (C,d) a *P-graded chain complex*, if C is a P-graded module in which each $C_p \subset C$ is a $\mathbb{Z}$-graded submodule of C. Since the chain complex C is P-graded, its boundary homomorphism d has a (P,P)-matrix. A partial order $\leq$ in P is called (C,d)-admissible, or briefly *d-admissible*, if d is a filtered homomorphism with respect to $(P,\leq)$, i.e., if

$$d_{pq} \neq 0 \;\Rightarrow\; p \leq q \tag{18}$$

for all $p,q \in P$. Note that a (C,d)-admissible partial order on P may not always exist. However, we have the following straightforward proposition and definition.

Proposition and Definition 5.7. *If a given P-graded chain complex (C, d) admits a (C, d)-admissible partial order on P, then the intersection of all such (C, d)-admissible partial orders on P is again a (C, d)-admissible partial order on P. We call it the* native *partial order of d.* $\square$

Definition 5.8. We say that the triple (P, C, d) is a *poset filtered chain complex* if (C, d) is a P-graded chain complex and the partial order in P is d-admissible.

Note that for a poset filtered chain complex (P, C, d) the module C is not only P-graded but also $\mathbb{Z}$-graded where the nth summand of the $\mathbb{Z}$-gradation is the direct sum over $p \in P$ of the nth summands in the $\mathbb{Z}$-gradation of C_p.

Example 5.9. Consider the set of words

$$X := \{\, \mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{AB}, \mathbf{AC}, \mathbf{BC}, \mathbf{CD}, \mathbf{CE}, \mathbf{ABC} \,\}$$

and the free module $C := \mathbb{Q}\langle X\rangle$ with basis X and coefficients in the field $\mathbb{Q}$ of rational numbers. For a word $x \in X$ define its *dimension* as one less than the number of characters in x and denote the set of words of dimension i by X_i. Setting

$$C_i := \begin{cases} \mathbb{Q}\langle X_i\rangle & \text{if } X_i \neq \varnothing, \\ 0 & \text{otherwise.} \end{cases}$$

we obtain a $\mathbb{Z}$-gradation of C given by

$$C = \bigoplus_{i \in \mathbb{Z}} C_i. \tag{19}$$

The family

$$\mathcal{A} := \{\, \{\mathbf{A}\}, \{\mathbf{B}\}, \{\mathbf{AB}\}, \{\mathbf{C}, \mathbf{AC}, \mathbf{BC}, \mathbf{ABC}\}, \{\mathbf{CD}\}, \{\mathbf{CE}\} \,\}$$

is a partition of X. It makes C an $\mathcal{A}$-graded module. Consider the partial order in $\mathcal{A}$ given by the Hasse diagram

$$\begin{tikzcd}
\{\mathbf{CD}\} \arrow[dr, dash] & & \{\mathbf{CE}\} \arrow[dl, dash] \\
& \{\mathbf{C}, \mathbf{AC}, \mathbf{BC}, \mathbf{ABC}\} \arrow[d, dash] & \\
& \{\mathbf{AB}\} \arrow[dl, dash] \arrow[dr, dash] & \\
\{\mathbf{A}\} & & \{\mathbf{B}\}
\end{tikzcd} . \tag{20}$$

and a homomorphism $d : C \to C$ defined on the basis X by the matrix

	A	**B**	**AB**	**C**	**AC**	**BC**	**ABC**	**CD**	**CE**
A			-1		-1				
B			1			-1			
AB							1		
C					1	1		-1	-1
AC							-1		
BC							1		
ABC									
CD									
CE									

One can check that the triple $(\mathcal{A}, C, d)$ with the $\mathbb{Z}$-gradation (19) and $\mathcal{A}$ partially ordered as in the Hasse diagram (20) is a well-defined poset filtered chain complex.

Note that if $J \subset P$, then (C_J, d_{JJ}) need not be a chain complex in general. However, we have the following proposition.

Proposition 5.10. *Assume that J is a convex subset of P. Then we have:*

(i) *(C_J, d_{JJ}) is a chain complex.*

(ii) *(C_J, d_{JJ}) is chain isomorphic to the quotient complex $(C_{J^\leq}/C_{J^<}, d')$ where d' denotes the homomorphism induced by $d_{J^\leq J^\leq}$. Notice that the quotient complex is well-defined since $J^<$ is a down set due to the convexity of J.*

Proof: In order to prove (i) we need to verify that $d_{JJ}^2 = 0$. Assume first that J is a down set. Let $x \in C_J$. Then Corollary 4.6 immediately implies that $dx \in C_J$. Hence, one obtains both $d_{JJ}x = (\iota_J \circ d \circ \pi_J)(x) = dx$ and $d_{JJ}^2 x = d_{JJ} dx = d^2 x = 0$, which yields $d_{JJ}^2 = 0$. If J is just convex, we consider the down sets $I := J^<$ and $K := J^\leq$, which clearly satisfy the identity $K = I \cup J$. Since d is a filtered homomorphism, we have $d_{pq} = 0$ for $p \in J$ and $q \in I$.

Therefore, the matrix of d_{KK} takes the form.

$$\begin{bmatrix} d_{II} & d_{IJ} \\ 0 & d_{JJ} \end{bmatrix}.$$

Since K is a down set, we already have verified that $d_{KK}^2 = 0$. It follows that

$$0 = d_{KK}^2 = \begin{bmatrix} d_{II}^2 & d_{II}d_{IJ} + d_{IJ}d_{JJ} \\ 0 & d_{JJ}^2. \end{bmatrix}$$

Thus, $d_{JJ}^2 = 0$, which proves (i).

To establish (ii), we consider again the down sets $I := J^<$ and $K := J^\leq$, as well as the homomorphism $\kappa : C_J \ni x \mapsto [x]_I \in C_K/C_I$, where $[x]_I$ denotes the equivalence class of x in the quotient module C_K/C_I. Let $x \in C_J$ and let $\bar{d}_{KK}$ denote the boundary homomorphism induced by d_{KK} on the quotient module C_K/C_I. We have $\bar{d}_{KK}\kappa x = [dx]_I = [d_{IJ}x + d_{JJ}x]_I =$

$[d_{JJ}x]_I = \kappa d_{JJ}x$, which proves that κ is a chain map. Assume that $[x]_I = 0$ for an $x \in C_J$. Then $x \in C_I \cap C_J = \{0\}$, that is, $x = 0$. This proves that κ is a monomorphism. Finally, given the class $[y]_I \in C_K/C_I$ generated by $y \in C_K$, we have $y = y_I + y_J$ for a $y_I \in C_I$ and a $y_J \in C_J$. It follows that $[y]_I = [y_J]_I = \kappa y_J$, proving that κ is an epimorphism. Hence, κ is an isomorphism. $\square$

Given a poset filtered chain complex (P, C, d) we consider the poset P as an object of DPSet with the distinguished subset given by

$$P_\star := \{\, p \in P \mid \; C_p \text{ is homotopically essential.}\, \}$$

Recall that C_p is homotopically essential if it is not chain homotopic to the zero chain complex. We would like to point out that the definition of $P_\star$ depends on C. In the rare cases when we simultaneously consider two poset filtered chain complexes with the same underlying poset P, we write $P_\star^C$ instead of $P_\star$ to indicate the dependence of $P_\star$ on C.

Given two poset filtered chain complexes (P, C, d) and (P', C', d') we say that the map $(\alpha, h) : (P, C, d) \to (P', C', d')$ is a *filtered chain morphism*, if $h : (C, d) \to (C', d')$ is a chain map and $(\alpha, h) : (P, C) \to (P', C')$ is a filtered morphism. We say that the map (α, h) is a *graded chain morphism* if $h : (C, d) \to (C', d')$ is a chain map and $(\alpha, h) : (P, C) \to (P', C')$ is a graded morphism.

We define the category PfCc of filtered chain complexes by taking poset filtered chain complexes as objects and filtered chain morphisms as morphisms. One easily verifies that this is indeed a category. We also define the subcategory PgCc of graded chain complexes by taking the same objects as in PfCc and graded chain morphisms as morphisms.

Definition 5.11. Given a pair of filtered chain morphisms $(\alpha, \varphi), (\alpha', \varphi') : (P, C, d) \to (P', C', d')$ we say that a filtered module morphism $(\gamma, \Gamma) : (P, C) \to (P', C')$ is an *elementary filtered chain homotopy* between (α, φ) and (α', φ') if the following conditions are satisfied:

(i) Γ is a degree $+1$ module homomorphism with respect to the $\mathbb{Z}$-gradation of C and C',
(ii) Γ is a chain homotopy between φ and φ', that is, $\varphi' - \varphi = \Gamma d + d'\Gamma$,
(iii) $\alpha_{|P'_\star} = \gamma_{|P'_\star} = \alpha'_{|P'_\star}$.

We say that two filtered chain morphisms (α, h) and (α', h') are *elementarily filtered chain homotopic*, and we write $(\alpha, h) \sim_e (\alpha', h')$, if there exists an elementary filtered chain homotopy between (α, h) and (α', h'). We say that filtered chain morphisms (α, h), (α', h') are *filtered chain homotopic*, and we write $(\alpha, h) \sim (\alpha', h')$, if there exists a sequence

$$(\alpha, h) = (\alpha_0, h_0) \;\sim_e\; (\alpha_1, h_1) \;\sim_e\; \dots \;\sim_e\; (\alpha_n, h_n) = (\alpha', h') \tag{21}$$

of filtered chain morphisms such that successive pairs are elementarily filtered chain homotopic.

The following proposition is straightforward.

Proposition 5.12. *The relation $\sim$ in the set of morphisms from (P, C, d) to (P', C', d') in* PfCc *is an equivalence relation.* $\square$

We say that a poset filtered chain complex (P, C, d) is *homotopically essential* if (C_p, d_{pp}) is homotopically essential for each $p \in P$, i.e., if $P_\star = P$.

Proposition 5.13. *Assume that the poset filtered chain complex (P', C', d') is homotopically essential. If (α, h) and $(\alpha', h') : (P, C, d) \to (P', C', d')$ are filtered chain homotopic, then $\alpha = \alpha'$, the domain of $\alpha = \alpha'$ is P', and we have in fact that $(\alpha, h) \sim_e (\alpha', h')$.*

Proof: Choose $n + 1$ filtered morphisms $(\alpha_i, h_i) : (P, C, d) \to (P', C', d')$ as in (21) for $i = 0, \ldots, n$. Let $(\gamma_i, \Gamma_i) : (P, C) \to (P', C')$ for $i \in \{1, 2, \ldots, n\}$ be an elementary filtered chain homotopy between (α_{i-1}, h_{i-1}) and (α_i, h_i). Since $P'_\star = P'$, we have $\alpha_{i-1} = (\alpha_{i-1})_{|P'_\star} = (\gamma_i)_{|P'_\star} = \gamma_i$, as well as the identity $\alpha_i = (\alpha_i)_{|P'_\star} = (\gamma_i)_{|P'_\star} = \gamma_i$ for $i \in \{1, 2, \ldots, n\}$. It follows then that one has $\alpha_i = \alpha_{i-1}$ for all i, and thus $\alpha = \alpha'$. Moreover, we obtain

$$h_i - h_{i-1} = d'\Gamma_i + \Gamma_i d \tag{22}$$

for $i \in \{1, 2, \ldots, n\}$. Let $\Gamma := \sum_{i=1}^{n} \Gamma_i$. One easily verifies that Γ is an α-filtered module homomorphism. Summing (22) for $i = 1, \ldots, n$ we get

$$h' - h = h_n - h_0 = d'\Gamma + \Gamma d.$$

Thus, (α, Γ) is an elementary filtered chain homotopy between the filtered morphisms (α, h) and (α', h'). $\square$

We refer to the equivalence classes of $\sim$ as the *homotopy equivalence classes.* We define the *homotopy category* of poset filtered chain complexes, denoted ChPfCc, by taking poset filtered chain complexes as objects, homotopy equivalence classes of filtered chain morphisms in PfCc as morphisms in ChPfCc, and use the formula

$$[(\beta, g)]_\sim \circ [(\alpha, h)]_\sim := [(\alpha \circ \beta, g \circ h)]_\sim \tag{23}$$

for the two filtered morphisms $(\alpha, h) \in$ PfCc$((P, C, d), (P', C', d'))$ and $(\beta, g) \in$ PfCc$((P', C', d'), (P'', C'', d''))$ as the definition of composition of morphisms in ChPfCc. Finally, equivalence classes of identities in PfCc are identities in ChPfCc.

Proposition 5.14. *Assume filtered chain morphisms*

$$(\alpha, h), (\alpha', h') : (P, C, d) \to (P', C', d')$$

and

$$(\beta, g), (\beta', g') : (P', C', d') \to (P'', C'', d'')$$

are filtered chain homotopic. Then the compositions

$$(\beta, g) \circ (\alpha, h), (\beta', g') \circ (\alpha', h') : (P, C, d) \to (P'', C'', d'')$$

are also filtered chain homotopic. In particular, the category ChPfCc *is well-defined.*

Proof: Assume that $(\alpha,h)\sim(\alpha',h')$ and $(\beta,g)\sim(\beta',g')$. Since $\sim$ is an equivalence relation, it suffices to prove that $(\beta,g)\circ(\alpha,h)\sim(\beta,g)\circ(\alpha',h')$ and $(\beta,g)\circ(\alpha',h')\sim(\beta',g')\circ(\alpha',h')$. For the same reason, we may assume that $(\alpha,h)\sim_e(\alpha',h')$ and $(\beta,g)\sim_e(\beta',g')$. Let $(\gamma,S):(P,C)\to(P',C')$ be an elementary filtered chain homotopy between (α,h) and (α',h'). Consider the filtered morphism $(\eta,T):=(\beta,g)\circ(\gamma,S):(P,C)\to(P'',C'')$. Then $\eta=\gamma\beta$ and $T=gS$. We will prove that (η,T) is an elementary filtered chain homotopy between $(\beta,g)\circ(\alpha,h)$ and $(\beta,g)\circ(\alpha',h')$. For this, we need to verify properties (i)-(iii) of Definition 5.11. Clearly, T is a degree $+1$ $\mathbb{Z}$-graded module homomorphism. Hence, property (i) is satisfied. To see property (ii), observe that

$$gh'-gh=g(h'-h)=g(d'S+Sd)=d''gS+gSd=d''T+Td.$$

Finally, in view of the inclusion $\beta(P''_\star)\subset P'_\star$, which is a consequence of the definition of DPSet, one obtains

$$\alpha\beta_{|P''_\star}=\alpha_{|P'_\star}\beta_{|P''_\star}=\gamma_{|P'_\star}\beta_{|P''_\star}=\eta_{|P''_\star}=\gamma_{|P'_\star}\beta_{|P''_\star}=\alpha'_{|P'_\star}\beta_{|P''_\star}=\alpha'\beta_{|P''_\star}.$$

Hence, we proved that $(\beta,g)\circ(\alpha,h)\sim_e(\beta,g)\circ(\alpha',h')$ which implies that also $(\beta,g)\circ(\alpha,h)\sim(\beta,g)\circ(\alpha',h')$. The proof that $(\beta,g)\circ(\alpha',h')\sim(\beta',g')\circ(\alpha',h')$ is similar. $\square$

We say that a filtered chain morphism $(\alpha,h):(P,C,d)\to(P',C',d')$ is a *filtered chain equivalence* if $[(\alpha,h)]$ is an isomorphism in ChPfCc. We say that two poset filtered chain complexes (P,C,d) and (P',C',d') are *filtered chain homotopic* if they are isomorphic in ChPfCc, i.e., if there exist two filtered chain morphisms $(\alpha,\varphi):(P,C,d)\to(P',C',d')$ and $(\alpha',\varphi'):(P',C',d')\to(P,C,d)$ such that $(\alpha',\varphi')\circ(\alpha,\varphi)$ is filtered chain homotopic to $\mathrm{id}_{(P,C)}$ and $(\alpha,\varphi)\circ(\alpha',\varphi')$ is filtered chain homotopic to $\mathrm{id}_{(P',C')}$. In this situation, the filtered chain morphisms (α,φ) and (α',φ') are referred to as *mutually inverse* filtered chain equivalences.

Definition 5.15. Let (P,C,d) be a poset filtered chain complex. By a *representation* of (P,C,d) we mean an object (P',C',d') in PfCc which is filtered chain homotopic to (P,C,d), together with mutually inverse filtered chain equivalences $(\alpha,\varphi):(P,C,d)\to(P',C',d')$ and $(\beta,\psi):(P',C',d')\to(P,C,d)$. More formally speaking, a representation of (P,C,d) is a triple $((P',C',d'),(\alpha,\varphi),(\beta,\psi))$. To simplify the terminology, when speaking about a representation we always assume that the associated mutually inverse chain equivalences are implicitly given.

Let $((P',C',d'),(\alpha,\varphi),(\beta,\psi))$ and $((P'',C'',d''),(\alpha',\varphi'),(\beta',\psi'))$ be two representations of a filtered chain complex (P,C,d).

Definition 5.16. We define the *transfer morphism* from the filtered chain complex (P',C',d') to the filtered chain complex (P'',C'',d'') as the filtered chain morphism $(\beta\alpha',\varphi'\psi)$.

Note that the pair of transfer morphisms $(\beta\alpha', \varphi'\psi)$ from (P', C', d') to (P'', C'', d'') and $(\beta'\alpha, \varphi\psi')$ from (P'', C'', d'') to (P', C', d') are mutually inverse chain equivalences.

We finish this section with the following auxiliary proposition.

Proposition 5.17. *Assume that (P, C, d) is a poset filtered chain complex with the P-gradation $(C_p)_{p\in P}$, and let $(W_p)_{p\in P}$ denote another P-gradation of C. If $(C_p)_{p\in P}$ and $(W_p)_{p\in P}$ are filtered equivalent in the sense of Definition 4.7, then the triple (P, W, d) with $W := \bigoplus_{p\in P} W_p$ and P-gradation given by $(W_p)_{p\in P}$ is also a poset filtered chain complex. Moreover, (P, W, d) and (P, C, d) are isomorphic in* ChPfCc.

Proof: Note that W and C are in fact the same modules, but the gradations $(W_p)_{p\in P}$ and $(C_p)_{p\in P}$ need not be the same. We know that d is a filtered homomorphism with respect to the $(C_p)_{p\in P}$ gradation of C. Since the $(W_p)_{p\in P}$ gradation of $W = C$ is filtered equivalent to the $(C_p)_{p\in P}$ gradation, we see from Proposition 4.8 that d is a filtered homomorphism with respect to the $(W_p)_{p\in P}$ gradation of C as well. Hence, (P, W, d) is a poset filtered chain complex. We will prove that

$$(\mathrm{id}_P, \mathrm{id}_C) : (P, C, d) \to (P, W, d) \tag{24}$$

and

$$(\mathrm{id}_P, \mathrm{id}_W) : (P, W, d) \to (P, C, d) \tag{25}$$

are mutually inverse isomorphisms in PfCc. Let $p \in P$. Since both $(C_p)_{p\in P}$ and $(W_p)_{p\in P}$ are filtered equivalent, using Proposition 5.10(ii) we get

$$C_p \cong C_{p\leq}/C_{p<} = W_{p\leq}/W_{p<} \cong W_p.$$

Hence, C_p is essential if and only if W_p is essential. It follows that $P^C_\star = P^W_\star$. In consequence, $\mathrm{id}_P : (P, P^C_\star) \to (P, P^W_\star)$ and $\mathrm{id}_P : (P, P^W_\star) \to (P, P^C_\star)$ are well-defined morphisms in DPSet. Since $(C_p)_{p\in P}$ and $(W_p)_{p\in P}$ are filtered equivalent, we get from Corollary 4.6 that $(\mathrm{id}_P, \mathrm{id}_C) : (P, C) \to (P, W)$ and $(\mathrm{id}_P, \mathrm{id}_W) : (P, W) \to (P, C)$ are filtered morphisms. Clearly, both are chain maps. It follows that the morphisms (24) and (25) are well-defined. Obviously, they are mutually inverse. Thus, the conclusion follows. □

6. Algebraic connection matrices

6.1. **Reduced filtered chain complexes.** Given a poset filtered chain complex (P, C, d), a singleton $\{p\} \subset P$ is always convex. Therefore, by Proposition 5.10(ii) the pair (C_p, d_{pp}) is a chain complex.

Definition 6.1. We say that a poset filtered chain complex (P, C, d) is *reduced* if (C_p, d_{pp}) is boundaryless and homotopically essential for all $p \in P$.

Example 6.2. The filtered chain complex in Example 5.9 is not reduced, because one can check that $C_{\{C,AC,BC,ABC\}}$ is inessential. Consider another set of words

$$X' := \{\, \mathbf{A}, \mathbf{B}, \mathbf{AB}, \mathbf{AD}, \mathbf{AE} \,\}.$$

Proceeding as in Example 5.9 we obtain a $\mathbb{Z}$-graded $\mathbb{Q}$-module C'. Consider the partial order in $\mathcal{A}' := \{\, \{x\} \mid x \in X' \,\}$ given by the Hasse diagram

(26)
$$\begin{tikzcd}
\{\mathbf{AB}\} \arrow[d, dash] \arrow[dr, dash] & \{\mathbf{AD}\} \arrow[d, dash] & \{\mathbf{AE}\} \arrow[dl, dash] \\
\{\mathbf{B}\} & \{\mathbf{A}\} &
\end{tikzcd}$$

and a homomorphism $d' : C' \to C'$ defined on the basis X' by the matrix

(27)

d'	**A**	**B**	**AB**	**AD**	**AE**
A			-1	-1	-1
B			1		
AB					
CD					
CE					

One can check that the triple $(\mathcal{A}', C', d')$, with $\mathcal{A}'$ partially ordered as in the Hasse diagram (20), is a well-defined poset filtered chain complex. It is reduced, because in each case exactly one chain group is nontrivial and the boundary operator is always zero within each C_p.

As an immediate consequence of Corollary 5.4 we get the following proposition.

Proposition 6.3. *A poset filtered chain complex (P, C, d) is reduced if and only if $d_{pp} = 0$ and $C_p \neq 0$ for all $p \in P$.*

Proposition 6.4. *Assume that (P, C, d) and (P', C', d') are two reduced poset filtered chain complexes and that $(\alpha, h), (\beta, g) : (P, C, d) \to (P', C', d')$ are filtered chain homotopic morphisms. If α is injective, then for $p \in P$ and $q \in P'$ we have*

$$\alpha(q) = p \;\Rightarrow\; h_{qp} = g_{qp}.$$

Proof: Since (P', C', d') is reduced, it is homotopically essential. Thus, we get from Proposition 5.13 that $\alpha = \beta : P' \to P$ and there exists an α-filtered degree $+1$ homomorphism $\Gamma : C \to C'$ such that $g - h = d'\Gamma + \Gamma d$. Since (P, C, d) and (P', C', d') are reduced, we get from Proposition 4.11 the identities

$$\begin{aligned}
g_{qp} - h_{qp} &= (d'\Gamma)_{qp} + (\Gamma d)_{qp} = d'_{q\,\mathrm{id}(q)}\Gamma_{\mathrm{id}(q)p} + \Gamma_{q\alpha(q)} d_{\alpha(q)p} \\
&= 0\Gamma_{qp} + \Gamma_{qp}0 = 0.
\end{aligned}$$

For this, recall also that d and d' are id_P-filtered, and that id_P is injective. This completes the proof. □

If filtered chain complexes (P, C, d) and (P', C', d') are filtered chain homotopic, then the equivalence class of every filtered chain equivalence $(\alpha, \varphi) : (P, C, d) \to (P', C', d')$ is automatically an isomorphism in ChPfCc. However, in the category PfCc one has to prove that fact. This is addressed in the following result.

Theorem 6.5. *Assume that the two poset filtered chain complexes (P, C, d) and (P', C', d') are reduced. If they are filtered chain homotopic, then every filtered chain equivalence $(\alpha, \varphi) : (P, C, d) \to (P', C', d')$ is an isomorphism in* PfCc. *In particular, (P, C, d) and (P', C', d') are isomorphic in* PfCc.

Proof: Assume that both (P, C, d) and (P', C', d') are reduced. Then we have $P_\star = P$ and $P'_\star = P'$. Let $(\alpha', \varphi') : (P', C', d') \to (P, C, d)$ be a filtered chain morphism such that one has both $(\alpha', \varphi') \circ (\alpha, \varphi) \sim \mathrm{id}_{(P,C,d)}$ and $(\alpha, \varphi) \circ (\alpha', \varphi') \sim \mathrm{id}_{(P',C',d')}$. Clearly, id_P and $\mathrm{id}_{P'}$ are injective. Hence, it follows from Proposition 6.4 that

$$(\varphi'\varphi)_{pp} = (\mathrm{id}_C)_{pp} = \mathrm{id}_{C_p} \quad \text{and} \quad (\varphi\varphi')_{qq} = (\mathrm{id}_{C'})_{qq} = \mathrm{id}_{C'_q}$$

for arbitrary $p \in P$ and $q \in P'$. Therefore, it follows from Corollary 4.12 that $(\mathrm{id}_C)_{pp} = \varphi'_{p\alpha'(p)}\varphi_{\alpha'(p)p}$ for $p \in P$. Similarly we obtain the identity $(\mathrm{id}_{C'})_{p'p'} = \varphi_{p'\alpha(p')}\varphi'_{\alpha(p')p'}$ for $p' \in P'$. Since α, α' are mutually inverse bijections, we may substitute $\alpha'(p)$ for p' and we get $(\mathrm{id}_{C'})_{\alpha'(p)\alpha'(p)} = \varphi_{\alpha'(p)p}\varphi'_{p\alpha'(p)}$ Hence, we conclude that $\varphi_{\alpha'(p)p}$ and $\varphi'_{p\alpha'(p)}$ are mutually inverse module homomorphisms. Thus, it follows from Lemma 4.13 that (α, φ) is an isomorphism in FMod. Since φ is a chain map, by Proposition 5.1 its inverse is also a chain map. Hence, (α, φ) is an isomorphism in PfCc, which proves that (P, C, d) and (P', C', d') are isomorphic in PfCc. □

6.2. **Essentially graded morphisms.**

Definition 6.6. Let (P, C, d) and (P', C', d') be poset filtered chain complexes. We call a filtered chain morphism $(\alpha, h) : (P, C, d) \to (P', C', d')$ *essentially graded*, if there is a graded chain morphism $(\beta, g) : (P, C, d) \to (P', C', d')$ which is filtered chain homotopic to (α, h).

Proposition and Definition 6.7. *Assume $(\alpha, h) : (P, C, d) \to (P', C', d')$ is an essentially graded chain equivalence and that both poset filtered chain complexes (P, C, d) and (P', C', d') are reduced. Then there exists a unique graded chain morphism $(\beta, g) : (P, C, d) \to (P', C', d')$ in the homotopy equivalence class $[(\alpha, h)]$. We call it the* graded representation *of (α, h).*

Proof: The existence of (β, g) follows directly from the definition of an essentially graded morphism. To prove uniqueness, we assume that (β, g) and (β', g') are two graded chain morphisms in the homotopy equivalence class $[(\alpha, h)]$. Since (α, h) is a chain equivalence, we can select a filtered chain morphism $(\alpha', h') : (P', C', d') \to (P, C, d)$ such that $(\alpha\alpha', h'h) \sim \mathrm{id}_{(P,C,d)}$ and $(\alpha'\alpha, hh') \sim \mathrm{id}_{(P',C',d')}$. Hence, it follows from Proposition 5.13 that $\alpha\alpha' = \mathrm{id}_P$ and $\alpha'\alpha = \mathrm{id}_{P'}$. In particular, α is injective. Since $(\beta, g) \sim (\alpha, h)$ and $(\beta', g') \sim (\alpha, h)$, we see that $(\beta, g) \sim (\beta', g')$ and from Proposition 5.13 we get $\beta = \alpha = \beta'$. Hence, β is injective. Consider $p \in P$ and $q \in P'$. If we have $p = \alpha(q)$, then one obtains $g_{qp} = g'_{qp}$ from Proposition 6.4. If $p \neq \alpha(q)$, we get $g_{qp} = 0 = g'_{qp}$ from (8), because both g and g' are α-graded. □

The following proposition is an immediate consequence of Proposition 5.14 and the fact that the composition of graded chain morphisms is again a graded chain morphism. The later statement in turn follows from the fact that the composition of graded morphisms is a graded morphism, and that the composition of chain maps is again a chain map. Thus, we have the following result.

Proposition 6.8. *Assume that both $(\alpha, h) : (P, C, d) \to (P', C', d')$ and $(\alpha', h') : (P', C', d') \to (P'', C'', d'')$ are filtered chain morphisms which are essentially graded, and which have the respective graded representations $(\beta, g) : (P, C, d) \to (P', C', d')$ and $(\beta', g') : (P', C', d') \to (P'', C'', d'')$. Then the composition $(\alpha', h') \circ (\alpha, h)$ is also essentially graded and its graded representation is given by $(\beta', g') \circ (\beta, g)$.* □

It is straightforward to observe that every identity morphism in PfCc is essentially graded, and that it is its own graded representation. Thus, in view of Proposition 6.8 we have a well-defined wide subcategory EgPfCc of PfCc whose morphisms are essentially graded morphisms in PfCc.

Proposition 6.9. *Assume that both (P, C, d) and (P', C', d') are reduced poset filtered chain complexes, and that both $(\alpha, h) : (P, C, d) \to (P', C', d')$ and $(\alpha', h') : (P', C', d') \to (P, C, d)$ are mutually inverse chain equivalences. Suppose further that (α, h) is essentially graded and has the graded representation $(\bar{\alpha}, \bar{h})$. Then (α', h') is essentially graded as well, $(\bar{\alpha}, \bar{h})$ is an isomorphism in PgCc, and $(\bar{\alpha}, \bar{h})^{-1}$ is the graded representation of (α', h').*

Proof: By Definition 6.6 we have $(\alpha, h) \sim (\bar{\alpha}, \bar{h})$. Hence, we get from Proposition 5.14 the equivalences

$$(\bar{\alpha}, \bar{h}) \circ (\alpha', h') \sim (\alpha, h) \circ (\alpha', h') \sim \mathrm{id}_{(P',C',d')}, \tag{28}$$

$$(\alpha', h') \circ (\bar{\alpha}, \bar{h}) \sim (\alpha', h') \circ (\alpha, h) \sim \mathrm{id}_{(P,C,d)}, \tag{29}$$

which imply that $(\bar{\alpha}, \bar{h})$ is a filtered chain equivalence. Now Theorem 6.5 shows that $(\bar{\alpha}, \bar{h})$ is an isomorphism in PfCc, and it follows that $(\bar{\alpha}, \bar{h})$ is an isomorphism in FMod. Thus, we get from Proposition 5.13 that $\bar{\alpha} = \alpha$ and from Lemma 4.13 that α is an order isomorphism. In consequence, the inverse of $(\bar{\alpha}, \bar{h}) = (\alpha, \bar{h})$ in PfCc takes the form $(\alpha^{-1}, \bar{h}^{-1})$ and, clearly, the inverse $\bar{h}^{-1}$ is α^{-1}-graded. Moreover, from (28) we get

$$(\alpha', h') = (\alpha^{-1}, \bar{h}^{-1}) \circ (\alpha, \bar{h}) \circ (\alpha', h') \sim (\alpha^{-1}, \bar{h}^{-1})$$

which proves that (α', h') is essentially graded with $(\alpha, \bar{h})^{-1} = (\bar{\alpha}, \bar{h})^{-1}$ as its graded representation. □

6.3. **Conley complex and connection matrices.** Throughout this section, let (P, C, d) denote a poset filtered chain complex. The following definition is the central concept of this paper.

Definition 6.10. By a *Conley complex* of (P, C, d) we mean every reduced representation of (P, C, d), that is, a poset filtered chain complex $(\bar{P}, \bar{C}, \bar{d})$

which is reduced and filtered chain homotopic to (P, C, d). The $(\bar{P}, \bar{P})$-matrix of the boundary homomorphism $\bar{d}$ is then called a *connection matrix* of the poset filtered chain complex (P, C, d).

As an immediate consequence of Theorem 6.5, we obtain the following two corollaries.

Corollary 6.11. *The Conley complex of a poset filtered chain complex is unique up to an isomorphism in* PfCc. *In particular, the transfer homomorphism between two Conley complexes of a given poset filtered chain complex is an isomorphism in* PfCc. □

Corollary 6.12. *If two poset filtered chain complexes are filtered chain homotopic, that is, isomorphic in* ChPfCc, *then their Conley complexes are isomorphic in* PfCc. □

Example 6.13. One can verify that the reduced filtered chain complex in Example 6.2 is a Conley complex of the filtered chain complex in Example 5.9 with the associated connection matrix (27). More precisely, consider the map $\alpha' : \mathcal{A}' \to \mathcal{A}$ given by

$$\alpha'(x) := \begin{cases} \{\mathbf{CD}\} & \text{if } x = \{\mathbf{AD}\}, \\ \{\mathbf{CE}\} & \text{if } x = \{\mathbf{AE}\}, \\ x & \text{otherwise} \end{cases}$$

and homomorphisms $h' : C \to C'$ and $g' : C' \to C$ given by the matrices

h'	**A**	**B**	**AB**	**C**	**AC**	**BC**	**ABC**	**CD**	**CE**
A	1			1					
B		1							
AB			1			−1			
AD								1	
AE									1

and

g'	**A**	**B**	**AB**	**AD**	**AE**
A	1				
B		1			
AB			1		
C					
AC				1	1
BC					
ABC					
CD				1	
CE					1

.

One can verify that

$$(\alpha', h') : (\mathcal{A}, C, d) \to (\mathcal{A}', C', d') \text{ and } ((\alpha')^{-1}, g') : (\mathcal{A}', C', d') \to (\mathcal{A}, C, d),$$

with the partial map $(\alpha')^{-1} : \mathcal{A} \nrightarrow \mathcal{A}'$ defined as the inverse relation of α', are mutually inverse filtered chain equivalences. This immediately implies that a Conley complex of the filtered chain complex $(\mathcal{A}, C, d)$ in Example 5.9 is given by the representation $((\mathcal{A}', C', d'), (\alpha', h'), ((\alpha')^{-1}, g'))$.

6.4. **Equivalence of Conley complexes.**

Definition 6.14. We say that two Conley complexes of a poset filtered chain complex are *equivalent* if the associated transfer homomorphism is essentially graded.

Note that the transfer homomorphism is always an isomorphism in PfCc. For equivalence, one needs in addition that it is essentially graded, i.e., that it is filtered chain homotopic to a graded chain isomorphism.

It follows from Proposition 6.8 that the equivalence of Conley complexes of a given poset filtered chain complex is indeed an equivalence relation. Two equivalent Conley complexes, as well as the associated connection matrices, are basically the same. This justifies the following definition.

Definition 6.15. A poset filtered chain complex has a *uniquely* determined Conley complex and connection matrix if any two of its Conley complexes are equivalent.

Note that two non-equivalent Conley complexes of a filtered chain complex may be graded-conjugate and the associated connection matrices may be graded similar. Thus, graded similarity of connection matrices is not sufficient for their uniqueness.

Example 6.16. The filtered chain complex in Example 5.9 does not have a uniquely determined Conley complex. To see this consider the set of words

$$X'' := \{\, \mathbf{A}, \mathbf{B}, \mathbf{AB}, \mathbf{BD}, \mathbf{BE} \,\}.$$

Proceeding as in Example 5.9 we obtain a $\mathbb{Z}$-graded $\mathbb{Q}$-module C''. Consider the partial order in $\mathcal{A}'' := \{\, \{x\} \mid x \in X'' \,\}$ given by the Hasse diagram

$$\begin{tikzcd}[arrows=dash] \{\mathbf{AB}\} & \{\mathbf{AD}\} & \{\mathbf{AE}\} \\ \{\mathbf{A}\} \arrow[u] & \{\mathbf{B}\} \arrow[ul] \arrow[u] \arrow[ur] & \end{tikzcd} \tag{30}$$

and a homomorphism $d'' : C'' \to C''$ defined on the basis X'' by the matrix

(31)

d''	**A**	**B**	**AB**	**BD**	**BE**
A			-1		
B			1	-1	-1
AB					
BD					
BE					

One can check that the triple $(\mathcal{A}'', C'', d'')$ with $\mathcal{A}''$ partially ordered as in the Hasse diagram (26) is a well-defined poset filtered chain complex and it is reduced. Moreover, we define the map $\alpha'' : \mathcal{A}'' \to \mathcal{A}$ by

$$\alpha''(x) := \begin{cases} \{\mathbf{CD}\} & \text{if } x = \{\mathbf{BD}\}, \\ \{\mathbf{CE}\} & \text{if } x = \{\mathbf{BE}\}, \\ x & \text{otherwise} \end{cases}$$

and homomorphisms $h'' : C \to C''$ and $g'' : C'' \to C$ via the matrices

h''	**A**	**B**	**AB**	**C**	**AC**	**BC**	**ABC**	**CD**	**CE**
A	1								
B		1		1					
AB			1		1				
BD								1	
BE									1

and

g''	**A**	**B**	**AB**	**BD**	**BE**
A	1				
B		1			
AB			1		
C					
AC					
BC				1	1
ABC					
CD				1	
CE					1

.

Then one can verify that $((\mathcal{A}'', C'', d''), (\alpha'', h''), ((\alpha'')^{-1}, g''))$ is another reduced representation, that is, a Conley complex of the filtered chain complex $(\mathcal{A}, C, d)$ presented in Example 5.9. We claim that this Conley complex is not equivalent to the Conley complex $(\mathcal{A}', C', d')$ presented in Example 6.13. To see this, assume to the contrary that the transfer homomorphism from $(\mathcal{A}', C', d')$ to $(\mathcal{A}'', C'', d'')$ is essentially graded. Observe that this transfer homomorphism is $(\gamma, h''g')$, where $\gamma = (\alpha')^{-1}\alpha'' : \mathcal{A}'' \to \mathcal{A}'$ is given by

$$\gamma(x) := \begin{cases} \{\mathbf{AD}\} & \text{if } x = \{\mathbf{BD}\}, \\ \{\mathbf{AE}\} & \text{if } x = \{\mathbf{BE}\}, \\ x & \text{otherwise} \end{cases}$$

and $h''g'$ has the matrix

$h''g'$	**A**	**B**	**AB**	**AD**	**AE**
A	1				
B		1			
AB			1	1	1
BD				1	
BE					1

.

Let (γ', f) be a graded representation of $(\gamma, h''g')$. Then $(\gamma', f) \sim (\gamma, h''g')$ and from Proposition 5.13 we get $\gamma' = \gamma$, as well as

$$h''g' - f = d''\Gamma + \Gamma d' \tag{32}$$

for some γ-filtered degree $+1$ homomorphism $\Gamma : C' \to C''$. Evaluating both sides of (32) at $\mathbf{AD}$ we get

$$\mathbf{AB} + \mathbf{BD} - f(\mathbf{AD}) = (d''\Gamma)(\mathbf{AD}) - \Gamma(\mathbf{A}) = -\Gamma(\mathbf{A}), \tag{33}$$

because $C'_2 = 0$. Since f is γ-graded and $\gamma(\mathbf{AB}) = \mathbf{AB}$, we see that the identity $\langle f(\mathbf{AD}), \mathbf{AB}\rangle = 0$ holds. Hence, from (33) one obtains

$$1 = \langle \mathbf{AB} + \mathbf{BD} - f(\mathbf{AD}), \mathbf{AB}\rangle = -\langle \Gamma(\mathbf{A}), \mathbf{AB}\rangle.$$

It follows that $\Gamma_{\mathbf{AB},\mathbf{A}} \neq 0$ and, since Γ is γ-filtered, we immediately obtain the inequality $\mathbf{AB} = \gamma(\mathbf{AB}) \leq \mathbf{A}$. However, we see from the Hasse diagram (20) of the poset $\mathcal{A}'$ that $\mathbf{A} < \mathbf{AB}$. This contradiction proves that the two Conley complexes $(\mathcal{A}', C', d')$ and $(\mathcal{A}'', C'', d'')$ of $(\mathcal{A}, C, d)$ are not equivalent. The associated connection matrices are the matrices of d' and d'' which are shown in (27) and (31), respectively. Note that despite the fact that these matrices are not equivalent as connection matrices of $(\mathcal{A}, C, d)$, they are graded similar. To see this define $\chi : \mathcal{A}'' \to \mathcal{A}'$ by

$$\chi(x) := \begin{cases} \{\mathbf{AD}\} & \text{if } x = \{\mathbf{BD}\}, \\ \{\mathbf{AE}\} & \text{if } x = \{\mathbf{BE}\}, \\ \{\mathbf{AB}\} & \text{if } x = \{\mathbf{AB}\}, \\ \{\mathbf{B}\} & \text{if } x = \{\mathbf{A}\}, \\ \{\mathbf{A}\} & \text{if } x = \{\mathbf{B}\}. \end{cases}$$

and $h : C' \to C''$ by the matrix

h	**A**	**B**	**AB**	**AD**	**AE**
A		1			
B	1				
AB			-1		
BD				1	
BE					1

.

Then one easily verifies that χ is an order preserving bijection, that (χ, h) is a graded chain isomorphism, and that we have

$$(\mathrm{id}_{\mathcal{A}''}, d'') \circ (\chi, h) = (\chi, h) \circ (\mathrm{id}_{\mathcal{A}'}, d').$$

This readily establishes the graded similarity.

6.5. **Existence of Conley complexes.** Although in our setting the poset in a poset filtered chain complex is not fixed as in [22] and [8], the existence proof of a Conley complex for a poset filtered chain complex in our sense can be adapted from the argument in [22, Theorem 8.1, Corollary 8.2]. For the sake of completeness, we present the details. We begin with a straightforward proposition and a technical lemma, the latter of which is a counterpart to [22, Theorem 8.1].

Proposition 6.17. *Assume that A and B are submodules of a module X such that $X = A + B$. If B' is a submodule of B satisfying $B = A \cap B \oplus B'$, then we have $X = A \oplus B'$.* $\square$

Lemma 6.18. *Assume that $P \neq \varnothing$ and that (P, C, d) is a poset filtered chain complex with field coefficients. Then there exist families $\{W_p\}_{p\in P}$, $\{V_p\}_{p\in P}$, $\{B_p\}_{p\in P}$, $\{H_p\}_{p\in P}$ of $\mathbb{Z}$-graded submodules of C satisfying the following properties:*

- (i) *$C = \bigoplus_{p\in P} W_p$ and $(W_p)_{p\in P}$ is a P-gradation of C which is filtered equivalent to the gradation $(C_p)_{p\in P}$,*
- (ii) *$W_p = V_p \oplus H_p \oplus B_p$ for $p \in P$,*
- (iii) *$d(V_p) \subset B_p$ and $d_{|V_p} : V_p \to B_p$ is a module isomorphism,*
- (iv) *$d_{pp}(H_p) = 0$ for $p \in P$,*
- (v) *for $H := \bigoplus_{p\in P} H_p$ we have $d(H) \subset H$ and $(P, H, d_{|H})$ is a filtered chain complex.*

Proof: Assume that (P, C, d) is a poset filtered chain complex. We proceed by induction on $n := \operatorname{card} P$. First assume that $n = 1$. Let p_* be the unique element of P. Then $C = C_{p_*}$. We set $W_{p_*} := C$. It follows that the P-gradations $(C_p)_{p\in P}$ and $(W_p)_{p\in P}$ are identical, and therefore they are filtered equivalent, i.e., (i) is satisfied. The existence of V_{p_*}, B_{p_*}, H_{p_*} satisfying properties (ii)-(v) follows from Proposition 5.5.

Now, assume that $n > 1$. Let $r \in P$ be a maximal element in P and consider the down set $P' := P \setminus \{r\} \in \operatorname{Down}(P)$. Let $C' := \bigoplus_{p\in P'} C_p$. Since d is a filtered homomorphism, we have $d(C') \subset C'$. Thus, $(P', C', d_{|C'})$ is a P'-filtered chain complex. Since $\operatorname{card} P' = n - 1$, by our induction hypothesis, there exist families $\{W_p\}_{p\in P'}$, $\{V_p\}_{p\in P'}$, $\{B_p\}_{p\in P'}$, $\{H_p\}_{p\in P'}$ of submodules such that properties (i)-(iv) hold for $(P', C', d_{|C'})$. In order to have respective families for (P, C, d) we will extend the families over P' to families over P by constructing in turn the modules V_r, B_r, H_r and W_r.

To begin with, for a family $\{M_p\}_{p\in P'}$ of submodules of C which satisfies the identity $M_p \cap M_q = \{0\}$ for all $p \neq q$, and an $L \in \operatorname{Down}(P')$, we introduce notation

$$M_L := \bigoplus_{p \in L} M_p.$$

Set $C' := C_{P'}$, $W' := W_{P'}$, $V' := V_{P'}$, $B' := B_{P'}$, and $H' := H_{P'}$. Then from (i) and (ii) applied to $(P', C', d_{|C'})$ we get

$$C'_L = C_L = W_L = V_L \oplus H_L \oplus B_L \quad \text{for every} \quad L \in \operatorname{Down}(P'). \tag{34}$$

We will show that

$$V' \cap (H' + d(C)) = 0. \tag{35}$$

Indeed, if $x \in V'$ and $x = x_1 + dx_2$ for an $x_1 \in H'$ and an $x_2 \in C$, then one obtains $dx = dx_1$. By (v) of the induction assumption, $(P', H', d_{|H'})$ is a poset filtered chain complex. Hence, $dx = dx_1 = d_{|H'}x_1 \in H'$. Also, by (iii) of the induction assumption we have $dx \in B'$, and therefore $dx \in H' \cap B' = 0$. Since $x \in V'$ one then obtains $x = 0$. This proves (35).

Since $d^{-1}(C_{r^<}) \cap C_{r^\leq}$ is a $\mathbb{Z}$-graded submodule of $C_{r^\leq}$, we can find a $\mathbb{Z}$-graded submodule V_r of $C_{r^\leq}$ such that

$$C_{r^\leq} = d^{-1}(C_{r^<}) \cap C_{r^\leq} \oplus V_r, \tag{36}$$

where we also use the fact that the modules have field coefficients. We will prove that

$$C_{r^\leq} \cap d^{-1}(C_{r^<}) = C_{r^<} + C_{r^\leq} \cap d^{-1}(H_{r^<}). \tag{37}$$

To see that the right-hand side of (37) is contained in the left-hand side observe that the right-hand side is obviously contained in $C_{r^\leq}$. Since C is a filtered chain complex and $r^<$ is a down set, we have $d(C_{r^<}) \subset C_{r^<}$ and, in consequence, $C_{r^<} \subset d^{-1}(C_{r^<})$. We also have $d^{-1}(H_{r^<}) \subset d^{-1}(C_{r^<})$, because $H_{r^<} \subset W_{r^<}$ by (ii) and $W_{r^<} = C_{r^<}$ by (i). To prove the opposite inclusion take an $x \in C_{r^\leq} \cap d^{-1}(C_{r^<})$. Then $dx \in C_{r^<}$. Hence, by (34), we can find $x_V \in V_{r^<}$, $x_H \in H_{r^<}$ and $x_B \in B_{r^<}$ such that $dx = x_V + x_H + x_B$. From (iii) we get $x_B = dy_V$ for some $y_V \in V_{r^<}$. It follows that $x_V = d(x - y_V) - x_H$. Hence, $x_V \in V_{r^<} \cap (d(C) + H_{r^<}) \subset V' \cap (d(C) + H')$. Thus, from (35) we get $x_V = 0$ and $d(x - y_V) = x_H \in H_{r^<}$ which means $x - y_V \in d^{-1}(H_{r^<})$. We also have $x \in C_{r^\leq}$ and $y_V \in V_{r^<} \subset C_{r^<} \subset C_{r^\leq}$. Therefore, one finally obtains $x = y_V + (x - y_V) \in C_{r^<} + C_{r^\leq} \cap d^{-1}(H_{r^<})$. This completes the proof of (37).

Now set $B_r := d(V_r)$. We will prove that

$$B_r \cap C_{r^<} = 0. \tag{38}$$

Let $x \in B_r \cap C_{r^<}$. Then $x = dy$ for some $y \in V_r \subset C_{r^\leq}$. Since $x \in C_{r^<}$, we get $y \in d^{-1}(C_{r^<})$. Therefore, $y \in V_r \cap C_{r^\leq} \cap d^{-1}(C_{r^<})$. It follows from (36) that $y = 0$. Hence, $x = d0 = 0$, which proves (38).

Since (P, C, d) is a poset filtered chain complex, the boundary homomorphism d is a filtered homomorphism. Therefore, we have $C_{r^<} \subset d^{-1}(C_{r^<})$.

Obviously, $C_{r<} \subset C_{r\leq}$ and $B_r = d(V_r) \subset d(C_{r\leq}) \cap d^{-1}(0) \subset C_{r\leq} \cap d^{-1}(H_{r<})$. Thus, by (38), we have a direct sum of $\mathbb{Z}$-graded submodules

$$C_{r<} \cap d^{-1}(H_{r<}) \oplus B_r \subset C_{r\leq} \cap d^{-1}(H_{r<}). \tag{39}$$

Hence, we can choose a $\mathbb{Z}$-graded submodule H_r such that

$$C_{r\leq} \cap d^{-1}(H_{r<}) = C_{r<} \cap d^{-1}(H_{r<}) \oplus B_r \oplus H_r. \tag{40}$$

Thus, it follows from (37) and Proposition 6.17 that

$$C_{r\leq} \cap d^{-1}(C_{r<}) = C_{r<} \oplus B_r \oplus H_r. \tag{41}$$

Therefore, setting $W_r := V_r \oplus B_r \oplus H_r$ we get from (36) that

$$C_{r\leq} = C_{r<} \oplus W_r. \tag{42}$$

We now have well-defined families $\{W_p\}_{p\in P}$, $\{V_p\}_{p\in P}$, $\{B_p\}_{p\in P}$, $\{H_p\}_{p\in P}$ of submodules of C. We will prove that they indeed satisfy properties (i)-(v) for (P, C, d). To prove (i) observe that by (42)

$$C = C_{P'\cup r^{\leq}} = C' + C_{r\leq} = C' + C_{r<} + W_r = C' + W_r.$$

We claim that $C = C' \oplus W_r$. Indeed, by (42) we have $W_r \subset C_{r\leq}$. Therefore, $W_r \cap C' \subset W_r \cap C_{r\leq} \cap C' = W_r \cap C_{r<} = 0$. This together with the induction assumption shows that

$$C = \bigoplus_{p\in P} W_p.$$

To show that $(W_p)_{p\in P}$ is filtered equivalent to $(C_p)_{p\in P}$, one needs to verify that $C_L = W_L$ for all $L \in \operatorname{Down}(P)$. Note that by the induction assumption

$$C_L = W_L \quad \text{for} \quad L \in \operatorname{Down}(P'). \tag{43}$$

Thus, we only need to consider the case when $r \in L$. Let $L' := L \setminus \{r\}$. Then $L = L' \cup r^{\leq}$, and (43) and (42) yield

$$C_L = C_{L'} + C_{r\leq} = W_{L'} + C_{r<} + W_r = W_{L'} + W_{r<} + W_r = W_{L'} + W_{r\leq} = W_L.$$

This proves property (i). By induction assumption properties (ii)-(v) need to be verified only for $p = r$. Property (ii) for $p = r$ follows from the definition of W_r. To see (iii) for $p = r$ take an $x \in V_r$ such that $dx = 0 \in C_{r<}$. Since the inclusion $V_r \subset C_{r\leq}$ holds, it follows that $x \in V_r \cap C_{r\leq} \cap d^{-1}(C_{r<})$ and we get from (36) that $x = 0$. Thus, $d_{|V_r}$ is a monomorphism. By the definition of B_r it is an epimorphism, which proves (iii).

Finally, by (40) we have $H_r \subset d^{-1}(H_{r<})$ which implies

$$d(H_r) \subset H_{r<}. \tag{44}$$

Therefore, $d_{rr}(H_r) = 0$ which proves (iv) for $p = r$. To show that $(P, H, d_{|H})$ is a poset filtered chain complex, we will prove that

$$d(H_L) \subset H_L \text{ for every } L \in \operatorname{Down}(P). \tag{45}$$

Property (45) holds by induction assumption if $r \notin L$. Thus, assume $r \in L$ and set $L' := L \setminus \{r\}$. Then $H_L = H_{L'} + H_{r^<} + H_r$, and by (44) and induction assumption one has

$$d(H_L) \subset d(H_{L'}) + d(H_{r^<}) + d(H_r) \subset H_{L'} + H_{r^<} \subset H_L,$$

which proves (45). Since $P \in \mathrm{Down}(P)$ and $H_P = H$, we get from (44) that the restriction $d_{|H} : H \to H$ is well-defined. Since $d^2 = 0$ and $d(H) \subset H$, we get $d_{|H}^2 = 0$. This proves that $(H, d_{|H})$ is a chain complex. From (45) and Corollary 4.6 one further obtains that $d_{|H}$ is a filtered homomorphism. This proves (v) and completes the proof of the lemma. □

Theorem 6.19. *Every poset filtered chain complex admits a Conley complex and a connection matrix.*

Proof: Assume (P, C, d) is a poset filtered chain complex. Let $\{W_p\}_{p\in P}$, $\{V_p\}_{p\in P}$, $\{B_p\}_{p\in P}$, $\{H_p\}_{p\in P}$ be families of submodules of C satisfying properties (i)-(v) of Lemma 6.18. Then, by Lemma 6.18(i), the collection $(W_p)_{p\in P}$ is a P-gradation of the module $W := \bigoplus_{p\in P} W_p$ coinciding with the module C and, by Proposition 5.17, the triple (P, W, d) is a filtered chain complex which is filtered chain isomorphic to (P, C, d). Thus, it suffices to prove that (P, W, d) admits a Conley complex and a connection matrix. For this, set $V := \bigoplus_{p\in P} V_p$, $B := \bigoplus_{p\in P} B_p$, $H := \bigoplus_{p\in P} H_p$. Then $W = V \oplus H \oplus B$. Moreover, (P, V), (P, B) and (P, H) are objects of GMod and by Lemma 6.18(iv) the triple $(P, H, d_{|H})$ is a filtered chain complex. Clearly, it is a filtered chain subcomplex of (P, W, d). Set $Q := \{\, p \in P \mid H_p \neq 0 \,\}$. We claim that $(Q, H, d_{|H})$ is also a filtered chain complex. Clearly, $H = \bigoplus_{p\in Q} H_p$ and $(H, d_{|H})$ is a chain complex. Let $L \in \mathrm{Down}(Q)$ and let $L' := \{\, p \in P \mid \exists_{q\in L}\, p \leq q \,\}$. Then $L' \in \mathrm{Down}(P)$ and $H_p = 0$ for $p \in L' \setminus L$. Therefore, $H_{L'} = H_L$. Thus, since $d_{|H}$ is P-filtered, it follows from Corollary 4.6 that $d_{|H}$ is also Q-filtered. Hence, $(Q, H, d_{|H})$ is a filtered chain complex. It follows from Lemma 6.18(iv) that $(Q, H, d_{|H})$ is boundaryless. Hence, we conclude from Corollary 5.4 and the definition of Q that $(Q, H, d_{|H})$ is reduced. We will prove that $(Q, H, d_{|H})$ is filtered chain homotopic to (P, W, d). Let $\iota : H \to W$ and $\pi : W \to H$ be respectively the inclusion and the projection homomorphisms. It follows from Lemma 6.18(v) that for $x \in H$ we have $d\iota x = dx = \iota dx$, that is, ι is a chain map. Also, for $x \in W$ we have $x = x_V + x_H + x_B$, where $x_V \in V$, $x_H \in H$ and $x_B \in B$. Hence, by Lemma 6.18(iii) $dx = dx_V + dx_H + dx_B = dx_V + dx_H \in B \oplus H$. Therefore, $\pi dx = dx_H = d\pi x$ proving that π is a chain map. Let $\alpha : Q \to P$ denote the inclusion map. Clearly, α is order preserving. Since α is injective, the inverse relation $\beta := \alpha^{-1} : P \nrightarrow Q$ is a well-defined partial map which is also order preserving. Recall that by Proposition 5.10(i) for every $p \in P$ we have a chain complex (W_p, d_{pp}). By Lemma 6.18 it has homology decomposition $W_p = V_p \oplus H_p \oplus B_p$. Thus, by Proposition 5.5(ii), the chain complexes

(W_p, d_{pp}) and $(H_p, 0)$ are chain homotopic. It follows that (W_p, d_{pp}) is essential if and only if $(H_p, 0)$ is essential. Moreover, by Corollary 5.4, $(H_p, 0)$ is essential if and only if $H_p \neq 0$. Therefore, $P_\star = Q = Q_\star$ which implies that $\alpha : (Q, Q_\star) \to (P, P_\star)$ and $\beta : (P, P_\star) \to (Q, Q_\star)$ are well-defined morphisms in DPSet. Obviously, the inclusion homomorphism $\iota : H \to W$ is β-graded and, in consequence, β-filtered. Similarly, the projection homomorphism $\pi : W \to H$ is α-filtered. Therefore, we have well-defined filtered morphisms $(\beta, \iota) : (Q, H, d_{|H}) \to (P, W, d)$ and $(\alpha, \pi) : (P, W, d) \to (Q, H, d_{|H})$. Obviously, $(\alpha, \pi) \circ (\beta, \iota) = (\beta\alpha, \pi\iota) = (\mathrm{id}_Q, \mathrm{id}_H) = \mathrm{id}_{(Q,H)}$. We will show that $(\beta, \iota) \circ (\alpha, \pi) = (\alpha\beta, \iota\pi)$ is filtered chain homotopic to $\mathrm{id}_{(P,W)}$. Let $\mu : W \ni x = x_V + x_H + x_B \mapsto x_B \in B$ be the projection map and let $\nu : V \ni x \mapsto x \in W$ be the inclusion map. Clearly, μ and ν are graded and, in consequence, filtered homomorphisms. By Lemma 6.18(iii) we have a well-defined P-graded degree -1 isomorphism $d_{|V} : V \ni x \mapsto dx \in B$ with a P-graded inverse, which is a degree $+1$ isomorphism $d_{|V}^{-1} : B \to V$. Then $\Gamma := \nu \circ d_{|V}^{-1} \circ \mu : W \to W$ is a degree 1 filtered module homomorphism. We claim that

$$\mathrm{id}_C - \iota\pi = \Gamma d + d\Gamma. \tag{46}$$

To see (46), take an arbitrary $x \in W = C$. Then $x = x_V + x_H + x_B$, where we have $x_V \in V$, $x_H \in H$ and $x_B \in B$. Hence, $(\mathrm{id}_C - \iota\pi)(x) = x_V + x_B$, as well as $dx = dx_V + dx_H \in B + H$, $\Gamma dx = x_V$, $\Gamma x = d_{|V}^{-1}(x_B)$ and also $d\Gamma x = x_B$. It follows that $(\Gamma d + d\Gamma)(x) = x_V + x_B$ which proves the identity (46). Clearly, the identity $(\beta\alpha)_{|P_\star} = \mathrm{id}_{P_\star} = \mathrm{id}_{P|P_\star}$ holds. Hence, (id_P, Γ) is an elementary filtered chain homotopy between $(\alpha\beta, \iota\pi)$ and $\mathrm{id}_{(P,W)}$. Thus, $(Q, H, d_{|H})$ and (P, W, d) are filtered chain homotopic. Since $Q = Q_\star$, we deduce from Lemma 6.18(iv) and Proposition 6.3 that $(Q, H, d_{|H})$ is reduced. It follows that the poset filtered chain complex $(Q, H, d_{|H})$ is in fact a Conley complex for (P, W, d) and that the (Q, Q)-matrix of $d_{|H}$ is a connection matrix of the poset filtered chain complex (P, W, d). □

7. Connection matrices in Lefschetz complexes

7.1. **Lefschetz complexes.** With this section we turn our attention to a more specialized situation. Rather than continuing to study connection matrices for general poset filtered chain complexes, we now consider the setting of Lefschetz complexes. The following definition goes back to S. Lefschetz, see [15, Chapter III, Section 1, Definition 1.1].

Definition 7.1. We say that (X, κ) is a *Lefschetz complex* over a ring R if $X = (X_q)_{q \in \mathbb{N}_0}$ is a finite set with $\mathbb{N}_0$-gradation, $\kappa : X \times X \to R$ is a map such that

$$\kappa(x, y) \neq 0 \quad \Rightarrow \quad x \in X_q, \;\; y \in X_{q-1}, \tag{47}$$

and for any $x, z \in X$ we have

$$\sum_{y \in X} \kappa(x, y)\kappa(y, z) = 0. \tag{48}$$

We refer to the elements of X as *cells*, to $\kappa(x, y)$ as the *incidence coefficient* of the cells x and y, and to κ as the *incidence coefficient map.* We define the *dimension* of a cell $x \in X_q$ as q, and denote it by $\dim x$. Whenever the incidence coefficient map is clear from the context we often refer just to X as Lefschetz complex. We say that (X, κ) is *regular* if for any $x, y \in X$ the incidence coefficient $\kappa(x, y)$ is either zero or it is invertible in R.

Let (X, κ) be a given Lefschetz complex. We denote by $C_k(X) := R\langle X_k\rangle$ the free R-module spanned by the set X_k of cells of dimension k for $k \in \mathbb{N}_0$, and let $C_k(X)$ denote the zero module for $k < 0$. Then it is clear that the sum $C(X) := \bigoplus_{k\in\mathbb{Z}} C_k(X)$ is a free $\mathbb{Z}$-graded R-module generated by X. Finally, define the module homomorphism $\partial^\kappa : C(X) \to C(X)$ on generators by

$$\partial^\kappa(x) := \sum_{y \in X} \kappa(x, y)y. \tag{49}$$

Proposition and Definition 7.2. *The pair $(C(X), \partial^\kappa)$ is a chain complex. We call it the* chain complex of (X, κ) *and we refer to the homology of this chain complex as the* Lefschetz homology *of (X, κ).*

Proof: Condition (47) guarantees that ∂^κ is a degree -1 module homomorphism, and condition (48) implies that $(\partial^\kappa)^2 = 0$. □

Note that every finitely generated free chain complex is the chain complex of a Lefschetz complex obtained by selecting a basis. More precisely, assume that (C, ∂) is a finitely generated free chain complex over a ring R and $U \subset C$ is a fixed basis of C. Suppose further that $C_k = 0$ for all $k < 0$. Then for every $v \in U$ there are uniquely determined coefficients $a_{vu} \in R$ such that

$$\partial v = \sum_{u \in U} a_{vu}u.$$

Let $\kappa_\partial : U \times U \to R$ be defined by $\kappa_\partial(v, u) = a_{vu}$. The following proposition is straightforward.

Proposition 7.3. *The pair (U, κ_∂) is a Lefschetz complex.* □

The family of cells of a simplicial complex [10, Definition 11.8] and the family of elementary cubes of a cubical set [10, Definition 2.9] provide simple but important examples of Lefschetz complexes. In these two cases the respective formulas for the incident coefficients are explicit and elementary, see for example [18]. Also a general regular cellular complex, or a regular finite CW complex as considered in [16, Section IX.3], is an example of a Lefschetz complex. In this case the incident coefficients may be obtained from a system of equations as described in [16, Section IX.5]. Note that a Lefschetz complex over a field is always regular.

Given $x, y \in X$ we say that y is a *facet* of x, and we write $y \prec_\kappa x$, if we have $\kappa(x, y) \neq 0$. It is easily seen that the reflexive and transitive closure of the relation $\prec_\kappa$ is a partial order. We denote this partial order by $\leq_\kappa$ and the associated strict order by $<_\kappa$. As an immediate consequence of (47) we have the following proposition.

Proposition 7.4. *The map* $\dim : (X, \leq_\kappa) \to (\mathbb{Z}, \leq)$ *which assigns any cell* $x \in X$ *its dimension* $\dim x \in \mathbb{Z}$ *is order preserving. Moreover, if the inequality* $x \prec_\kappa y$ *holds, then one has* $\dim y = \dim x + 1$.

We say that y is a *face* of x if $y \leq_\kappa x$. The T_0 topology defined via Theorem 3.5 by the partial order $\leq_\kappa$ will be called the *Lefschetz topology* of the Lefschetz complex (X, κ). Observe that the closure of a set $A \subset X$ in this topology consists of all faces of all cells in A.

We say that the subset $A \subset X$ is a *κ-subcomplex* or *Lefschetz subcomplex* of X, if $(A, \kappa_{|A \times A})$ with the $\mathbb{Z}$-gradation induced from X is a Lefschetz complex in its own right. The following proposition provides sufficient conditions for a subset of a Lefschetz complex to be a Lefschetz subcomplex. For more details see [17, Proposition 5.3] and [18, Theorem 3.1].

Proposition 7.5. *If* $A \subset X$ *is locally closed in the Lefschetz topology, then* A *is a Lefschetz subcomplex of* (X, κ)*. In particular, every open and every closed subset of* X *is a Lefschetz subcomplex.*

The following result was established in [17, Proposition 5.2], and it computes the homology of some simple Lefschetz complexes.

Proposition 7.6. *For every* $x \in X$ *the singleton* $\{x\}$ *is a Lefschetz subcomplex of* X *and*

$$H_q(\{x\}) \cong \begin{cases} R & \textit{if } q = \dim x, \\ 0 & \textit{otherwise.} \end{cases}$$

For every $x, y \in X$ *such that* x *is a facet of* y *the doubleton* $\{x, y\}$ *is a Lefschetz subcomplex of* X *and for all* $q \in \mathbb{Z}$ *one has*

$$H_q(\{x, y\}) \cong 0.$$

Note that in general the chain complex of a Lefschetz subcomplex A of a Lefschetz complex X is not a chain subcomplex of the chain complex of X. However, we have the following proposition.

Proposition 7.7. *If* A *is closed in* X *in the Lefschetz topology, then*

$$\partial^{\kappa_{|A\times A}} = \partial^\kappa_{|C(A)} \tag{50}$$

and

$$\partial^\kappa(C(A)) \subset C(A). \tag{51}$$

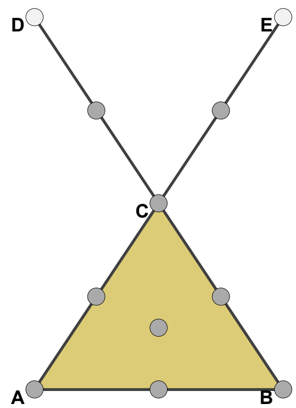

FIGURE 4. A simplicial complex consisting of the triangle **ABC**, the edges **AB**, **AC**, **BC**, **CD**, **CE**, and the vertices **A**, **B**, **C**, **D**, and **E** is an example of a Lefschetz complex. Simplices are marked with a small circle in the center of mass of each simplex. The simplices marked with a gray circle constitute a locally closed (convex) subset of the set of all simplices. Therefore, it is another example of a Lefschetz complex X which consists of all simplices of the simplicial complex except vertices **D** and **E**.

In particular, the chain complex $(C(A), \partial^{\kappa_{|A\times A}})$ *is a chain subcomplex of the chain complex* $(C(X), \partial^\kappa)$.

Proof: To see (50) it suffices to verify the equality on basis elements. Thus, take an $x \in A$. Then

$$\partial^{\kappa_{|A\times A}} x = \sum_{y\in A} \kappa(x,y)y = \sum_{y\in X} \kappa(x,y)y = \partial^\kappa x,$$

because $\kappa(x,y) \neq 0$ implies $y \in \operatorname{cl} x \subset \operatorname{cl} A = A$. Property (51) and the remaining assertion are obvious. □

Given a closed subset $A \subset X$ in the Lefschetz topology we define the relative Lefschetz homology $H(X, A)$ as the homology of the quotient chain complex $(C(X, A), \tilde{\partial})$, where $C(X,A) := C(X)/C(A)$ and $\tilde{\partial}$ stands for the induced boundary map.

Example 7.8. The easiest way to visualize a Lefschetz complex is by presenting it as a κ-subcomplex of a simplicial complex. A sample Lefschetz complex is presented in Figure 4 as a locally closed collection of simplices of a simplicial complex. Note, however, that not all Lefschetz complexes can be written in this form.

7.2. **Acyclic partitions.** Let X be a finite topological space and let $\mathcal{E}$ be a partition of X into locally closed sets. Consider the relation $\preceq_{\mathcal{E}}$ in $\mathcal{E}$ defined by

$$E \preceq_{\mathcal{E}} E' \quad \text{if and only if} \quad E \cap \operatorname{cl} E' \neq \varnothing. \tag{52}$$

We say that $\mathcal{E}$ is an *acyclic partition* of X if $\preceq_{\mathcal{E}}$ may be extended to a partial order on $\mathcal{E}$. Every such extension is called $\mathcal{E}$-admissible. Note that the smallest $\mathcal{E}$-admissible partial order on an acyclic partition $\mathcal{E}$ is the transitive closure of $\preceq_{\mathcal{E}}$. We call it the *inherent* partial order of $\mathcal{E}$ and denote it by $\leq_{\mathcal{E}}$.

Proposition 7.9. *Assume that $\mathcal{E}$ is an acyclic partition of a finite topological space X, that $\leq$ is an $\mathcal{E}$-admissible partial order, and that $\mathcal{D} \subset \mathcal{E}$.*

(i) *If $\mathcal{D}$ is a down set with respect to $\leq_{\mathcal{E}}$, then $|\mathcal{D}|$ is closed.*
(ii) *If $\mathcal{D}$ is convex with respect to $\leq_{\mathcal{E}}$, then $|\mathcal{D}|$ is locally closed.*

Proof: In order to prove (i), we assume that $\mathcal{D} \subset \mathcal{E}$ is a down set and let $x \in \operatorname{cl} |\mathcal{D}|$. Then there exists an $E' \in \mathcal{D}$ such that $x \in \operatorname{cl} E'$. Since $\mathcal{E}$ is a partition of X, there exists an $E \in \mathcal{E}$ such that $x \in E$. It follows that $E \cap \operatorname{cl} E' \neq \varnothing$ and, since $\leq$ is an $\mathcal{E}$-admissible partial order, we get $E \leq E'$. Since $\mathcal{D}$ is a down set, this in turn implies $E \in \mathcal{D}$. Thus, $x \in E \subset |\mathcal{D}|$, and therefore $|\mathcal{D}|$ is closed. This completes the proof of (i). To prove (ii), we now assume that $\mathcal{D}$ is convex. It follows from Proposition 3.1 that both $\mathcal{D}^{\leq}$ and $\mathcal{D}^{<}$ are down sets, and due to (i) the sets $|\mathcal{D}^{\leq}|$ and $|\mathcal{D}^{<}|$ are closed. Moreover, since $\mathcal{D} = \mathcal{D}^{\leq} \setminus \mathcal{D}^{<}$ and $\mathcal{E}$ is a partition, we have $|\mathcal{D}| = |\mathcal{D}^{\leq}| \setminus |\mathcal{D}^{<}|$. Hence, $|\mathcal{D}|$ is locally closed by Proposition 3.4. $\square$

7.3. **Connection matrices of acyclic partitions.** Assume that $\mathcal{E}$ is an acyclic partition of a Lefschetz complex X. Then the module $C(X)$ admits an $\mathcal{E}$-gradation

$$C(X) = \bigoplus_{E \in \mathcal{E}} C(E). \tag{53}$$

We have the following proposition.

Proposition 7.10. *The triple $(\mathcal{E}, C(X), \partial^{\kappa})$ with an $\mathcal{E}$-admissible partial order $\leq$ on $\mathcal{E}$ and the $\mathcal{E}$-gradation* (53) *is a poset filtered chain complex.*

Proof: We need to prove that ∂^{κ} is a filtered module homomorphism, that is, ∂^{κ} is an α-filtered module homomorphism with $\alpha = \mathrm{id}_{\mathcal{E}}$. For this, we will verify (11) of Corollary 4.6 for $M = C(X)$ and $L = \mathcal{D} \in \operatorname{Down}(\mathcal{E})$. Then one immediately obtains $M_L = C(X)_{\mathcal{D}} = C(|\mathcal{D}|)$, and we need to verify that the inclusion $\partial^{\kappa}(C(|\mathcal{D}|)) \subset C(|\mathcal{D}|)$ holds. But, this follows from Proposition 7.7, because, by Proposition 7.9(i), the set $|\mathcal{D}|$ is closed. Thus, by Corollary 4.6, we see that the boundary homomorphism is $\mathrm{id}_{\mathcal{E}}$-filtered. Therefore, $(\mathcal{E}, C(X), \partial^{\kappa})$ is a poset filtered chain complex. $\square$

Proposition 7.10 lets us define the Conley complex and connection matrices of acyclic partitions.

Definition 7.11. By a *filtration* of an acyclic partition $\mathcal{E}$ of a Lefschetz complex X we mean a poset filtered chain complex $(\mathcal{E}, C(X), \partial^\kappa)$ with $\mathcal{E}$ ordered by an $\mathcal{E}$-admissible partial order. The *Conley complex* and the associated *connection matrices* of $\mathcal{E}$ are defined as the Conley complex and the associated connection matrices of the filtration of $\mathcal{E}$, that is, the Conley complex and connection matrices of $(\mathcal{E}, C(X), \partial^\kappa)$.

7.4. **Singleton partition.** We have the following proposition and definition.

Proposition and Definition 7.12. *Assume (X, κ) is a Lefschetz complex. Then the following hold:*

(i) *The family*
$$\mathcal{X} := \{\, \{x\} \mid x \in X \,\}$$
is a partition of X into locally closed sets.

(ii) *The map*
$$\operatorname{sing} : X \ni x \mapsto \{x\} \in \mathcal{X}$$
is a bijection which preserves in both directions the relation $\leq_\kappa$ on X and the relation $\preceq_\mathcal{X}$ on $\mathcal{X}$ given by (52)*.*

(iii) *The relation $\preceq_\mathcal{X}$ is a partial order on $\mathcal{X}$ which coincides with the inherent partial order $\leq_\mathcal{X}$ on $\mathcal{X}$.*

(iv) *The family $\mathcal{X}$ is an acyclic partition of X.*

We call $\mathcal{X}$ the singleton partition *of X.*

Proof: Clearly $\mathcal{X}$ is a partition of X and every singleton $\{x\}$ is locally closed, because we have $\{x\} = x^{\leq_\kappa} \setminus x^{<_\kappa}$ and the sets $x^{\leq_\kappa}$ and $x^{<_\kappa}$ are both down sets, hence closed. This proves (i).

Now let $x, y \in X$. Then one easily verifies the sequence of equivalences

$$\begin{aligned} x \leq_\kappa y \quad &\Leftrightarrow \quad x \in \operatorname{cl} y \\ &\Leftrightarrow \quad \{x\} \cap \operatorname{cl}\{y\} \neq \varnothing \\ &\Leftrightarrow \quad \{x\} \preceq_\mathcal{X} \{y\}, \end{aligned}$$

which proves (ii). Hence, $\preceq_\mathcal{X}$ is a partial order on $\mathcal{X}$. In particular, $\preceq_\mathcal{X}$ is transitively closed, and this implies that $\preceq_\mathcal{X}$ coincides with the inherent partial order on $\mathcal{X}$. This proves (iii). It follows that $\mathcal{X}$ is an acyclic partition of X, which finally establishes (iv). □

Proposition 7.12 shows that we have a well-defined poset filtered chain complex $(\mathcal{X}, C(X), \partial^\kappa)$. Using the order isomorphism $\operatorname{sing} : X \to \mathcal{X}$ we identify it with $(X, C(X), \partial^\kappa)$. We note that via this identification closed sets in X are in one-to-one correspondence with down sets in $\mathcal{X}$.

Proposition 7.13. *Let (X, κ) be a Lefschetz complex. The native partial order of ∂^κ is precisely $\leq_\kappa$.*

Proof: According to Definition 5.7 we have to prove that for every admissible partial order $\leq$ on X and $x, y \in X$ the inequality $x \leq_\kappa y$ implies the inequality $x \leq y$. Clearly, it suffices to prove that $x \prec_\kappa y$ implies $x \leq y$, because $\leq_\kappa$ is the transitive closure of $\prec_\kappa$. But, $x \prec_\kappa y$ by definition means that $\kappa(y,x) \neq 0$ which implies $\partial^\kappa_{xy} \neq 0$, and by the admissibility of $\leq$ finally gives the inequality $x \leq y$. $\square$

Note that the bijection sing lets us identify partial orders in X with partial orders in the associated singleton partition $\mathcal{X}$ of X. As an immediate consequence of Proposition 7.12 and Proposition 7.13 we get the following corollary.

Corollary 7.14. *Let (X,κ) be a Lefschetz complex and let $\mathcal{X}$ denote the associated singleton partition. A partial order is ∂^κ-admissible if and only if it is $\mathcal{X}$-admissible.*

We say that a ∂^κ-admissible partial order $\leq$ in X is *natural* if

$$x \leq y \quad \text{and} \quad \dim x = \dim y \quad \Rightarrow \quad x = y. \tag{54}$$

Note that the native partial order of ∂^κ is natural, but that a ∂^κ-admissible partial order $\leq$ in X does not need to be natural in general.

Definition 7.15. A *filtration of a Lefschetz complex* (X,κ) is defined as a filtration of the singleton partition of X, that is, the poset filtered chain complex $(X, C(X), \partial^\kappa)$ with X ordered by a ∂^κ-admissible partial order. When we consider X as a poset ordered by a natural partial order in X we refer to the filtration $(X, C(X), \partial^\kappa)$ as a *natural filtration of* X. When we consider X as a poset ordered by the native partial order of X we refer to the filtration $(X, C(X), \partial^\kappa)$ as the *native filtration of* X.

Theorem 7.16. *Let (X,κ) be a Lefschetz complex. Then the following hold:*

(i) *Every filtration of the Lefschetz complex (X,κ) is reduced, that is, it is a Conley complex of itself. In particular, $X_\star = X$.*
(ii) *Every natural filtration of a Lefschetz complex (X,κ) has a uniquely determined Conley complex and connection matrix. The connection matrix coincides with the (X,X)-matrix of the boundary homomorphism ∂^κ.*

Proof: Consider an $x \in X$. We have from (47) that $\kappa_{|\{x\}\times\{x\}} = 0$. Therefore, $\partial^\kappa_{|C(\{x\})} = 0$. Moreover, since $C_{\dim x}(\{x\}) = Rx \neq 0$, we see from Proposition 6.3 that $X_\star = X$ and $(X, C(X), \partial^\kappa)$ is reduced. In consequence, it is a Conley complex of itself. This proves (i).

In order to prove (ii), it suffices to verify that every transfer morphism from $(X, C(X), \partial^\kappa)$ to a Conley complex (P, C, d) of $(X, C(X), \partial^\kappa)$ is essentially graded. Actually, we will prove the stronger fact that every such transfer morphism is graded. Thus, assume that $(X, C(X), \partial^\kappa)$ is a natural filtration of X, that (P, C, d) is another Conley complex of X, and that the map $(\alpha, \varphi) : (X, C(X), \partial^\kappa) \to (P, C, d)$ is a filtered chain isomorphism.

Since in view of (i) the filtration $(X, C(X), \partial^\kappa)$ is a Conley complex of itself, the transfer morphism from $(X, C(X), \partial^\kappa)$ to (P, C, d) is just (α, φ).

Since both $(X, C(X), \partial^\kappa)$ and (P, C, d), as Conley complexes, are reduced, we see from Proposition 6.9 that (α, φ) is also an isomorphism in FMod. Thus, it follows from Lemma 4.13 that $\alpha : P \to X$ is an order isomorphism and $\varphi_{p\alpha(p)} : C(\{\alpha(p)\}) \to C_p$ is an isomorphism of $\mathbb{Z}$-graded moduli for every $p \in P$. But, Proposition 7.6 provides the structure of $C(\{\alpha(p)\})$. Hence, we can choose a non-zero $c_p \in C$ such that

$$(C_p)_n = \begin{cases} Rc_p & \text{if } n = \dim \alpha(p), \\ 0 & \text{otherwise.} \end{cases}$$

In other words, C_p is a one-dimensional $\mathbb{Z}$-graded module generated by c_p such that

$$\dim c_p = \dim \alpha(p). \tag{55}$$

In order to prove that (α, φ) is an isomorphism in GMod we will first show that φ is α-graded by verifying (6). Assume therefore that we have $\varphi_{py} \neq 0$ for $p \in P$ and $y \in X$. Furthermore, let $x = \alpha(p)$. Since (α, φ) is a filtered module homomorphism, we get from (7) that $p \in \alpha^{-1}(y^{\leq})^{\leq}$. This implies that there exists a $p' \in P$ such that $p \leq p'$ and $\alpha(p') \leq y$. Since α is an order isomorphism, one further obtains $x = \alpha(p) \leq \alpha(p') \leq y$.

Note that C_p is a one-dimensional $\mathbb{Z}$-graded module which is non-zero only in dimension $n = \dim \alpha(p) = \dim x$. Since $\varphi_{py} : C(\{y\}) \to C_p$ is $\mathbb{Z}$-graded homomorphism of degree zero, both $C(\{y\})$ and C_p are one-dimensional $\mathbb{Z}$-graded modules, and we have $\varphi_{py} \neq 0$, one immediately obtains that φ_{py} maps Ry isomorphically onto Rc_p. It follows that $\dim c_p = \dim y$. In view of $x = \alpha(p)$ and (55) we therefore have $\dim x = \dim y$. Since $x \leq y$ and the partial order is natural, this implies $x = y$, as well as $\alpha(p) = x = y$. This in turn proves that φ is α-graded, that is, (α, φ) is a module homomorphism in GMod. Since it is an isomorphism in FMod, one obtains from Corollary 4.14 that it is also an isomorphism in GMod. Since it is a chain map, Proposition 5.1 shows that it is an isomorphism in PgCc. This completes the proof of (ii). □

8. Dynamics of combinatorial multivector fields

8.1. **Combinatorial multivector fields.** The concept of a combinatorial multivector field was proposed in [17, Definition 5.10] as a dynamically oriented extension of the notion of combinatorial vector field which has been introduced by Forman [6, 5]. The definition of combinatorial multivector field we consider here is based on [13] restricted to the special setting of Lefschetz complexes. Let X be a Lefschetz complex. By a *combinatorial multivector* in X we mean a non-empty subset of X which is locally closed with respect to the Lefschetz topology. A *combinatorial multivector field* is a partition $\mathcal{V}$ of X into combinatorial multivectors. Since in this paper we do not use any other concepts of vector fields, in the sequel we simplify the terminology by dropping the adjective combinatorial in combinatorial multivector and combinatorial multivector field.

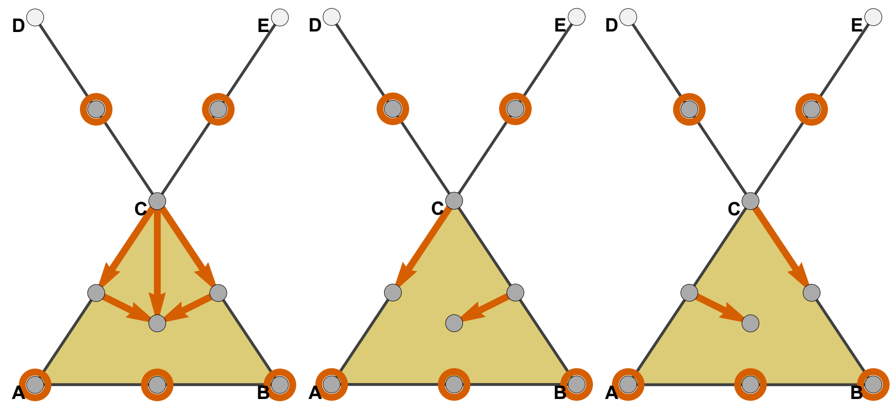

FIGURE 5. Three different multivector fields $\mathcal{V}_0$ (left), $\mathcal{V}_1$ (middle), $\mathcal{V}_2$ (right) on the Lefschetz complex X introduced in Example 7.8. Critical cells are marked with a fat circle around the circle in the center of mass of a simplex. Vectors and multivectors are marked with an arrow from simplex x to simplex y whenever $y \in \Pi_{\mathcal{V}}(x)$ and $y \notin \operatorname{cl} x$. The case when $y \in \operatorname{cl} x$ is not marked to keep the image readable.

We say that a multivector V is *critical*, if the relative Lefschetz homology $H(\operatorname{cl} V, \operatorname{mo} V)$ is non-zero. A multivector V which is not critical is called *regular*. For each $x \in X$ we denote by $[x]_{\mathcal{V}}$ the unique multivector in $\mathcal{V}$ which contains x. If the multivector field $\mathcal{V}$ is clear from context, we abbreviate the notation by writing $[x] := [x]_{\mathcal{V}}$. We say that $x \in X$ is *critical* (respectively *regular*) with respect to the multivector field $\mathcal{V}$, if the multivector $[x]_{\mathcal{V}}$ is critical (respectively regular). We say that a subset $A \subset X$ is $\mathcal{V}$-*compatible* if for each $x \in X$ either $[x]_{\mathcal{V}} \cap A = \varnothing$ or $[x]_{\mathcal{V}} \subset A$.

We associate with every multivector field a multivalued map $\Pi_{\mathcal{V}} : X \multimap X$ given by

$$\Pi_{\mathcal{V}}(x) := \operatorname{cl} x \cup [x]_{\mathcal{V}}.$$

The multivalued map $\Pi_{\mathcal{V}}$ may be interpreted as a digraph with vertices in X and an arrow from $x \in X$ to $y \in X$ if $y \in \Pi_{\mathcal{V}}(x)$. Clearly, every multivector $V \in \mathcal{V}$ forms a clique in this digraph. By collapsing all vertices in a multivector to a point we obtain an induced digraph with vertices in $\mathcal{V}$ and an arrow from $V \in \mathcal{V}$ to $W \in \mathcal{W}$ if there exist an $x \in V$ and a $y \in W$ such that $y \in \Pi_{\mathcal{V}}(x)$. We refer to this digraph as the $\mathcal{V}$-digraph.

Since $\mathcal{V}$ is a partition of X into locally closed subsets, we can consider the relation $\preceq_{\mathcal{V}}$ in $\mathcal{V}$ introduced in Section 7.2 . The following proposition shows that the $\mathcal{V}$-digraph and the relation $\preceq_{\mathcal{V}}$ are the same concepts.

Proposition 8.1. *There is an arrow from V to W in the $\mathcal{V}$-digraph of $\mathcal{V}$ if and only if $W \preceq_{\mathcal{V}} V$, that is, if and only if $W \cap \operatorname{cl} V \neq \varnothing$.*

Proof: Assume there is an arrow from V to W in the $\mathcal{V}$-digraph of $\mathcal{V}$, and let $x, y \in X$ be such that $x \in V$, $y \in W$, and $y \in \Pi_{\mathcal{V}}(x) = [x]_{\mathcal{V}} \cup \operatorname{cl} x$. If $y \in [x]$, then $W = [y] = [x] = V$ which yields $W \cap \operatorname{cl} V = V \neq \varnothing$. If $y \in \operatorname{cl}[x]$, then $y \in W \cap \operatorname{cl} V$ proving that $W \cap \operatorname{cl} V \neq \varnothing$. Vice versa, if $W \cap \operatorname{cl} V \neq \varnothing$, then we can take a $y \in W \cap \operatorname{cl} V$ and an $x \in V$ such that $y \in \operatorname{cl} x$. It follows that $y \in \Pi_{\mathcal{V}}(x)$. Hence, there is an arrow in the $\mathcal{V}$-digraph from V to W. $\square$

Example 8.2. Figure 5 presents three different combinatorial multivector fields $\mathcal{V}_0$, $\mathcal{V}_1$, and $\mathcal{V}_2$ on the Lefschetz complex X introduced in Example 7.8. The $\mathcal{V}_0$-digraph coincides with the Hasse diagram (20). Notice that by using the Hasse diagram representation of the digraph, we implicitly assume that arrows always point downwards, i.e., we represent them without arrow heads. Moreover, to keep the diagrams as simple as possible, we do not indicate the loops which are present at every node. Similarly, the $\mathcal{V}_1$-digraph is

$$\begin{tikzcd}
\{\mathbf{CD}\} \arrow[dr, dash] & & \{\mathbf{CE}\} \arrow[dl, dash] & \\
& \{\mathbf{C}, \mathbf{AC}\} \arrow[d, dash] & & \{\mathbf{BC}, \mathbf{ABC}\} \arrow[dll, dash] \\
& \{\mathbf{AB}\} \arrow[dl, dash] \arrow[dr, dash] & & \\
\{\mathbf{A}\} & & \{\mathbf{B}\} &
\end{tikzcd} \tag{56}$$

and the $\mathcal{V}_2$-digraph is given by

$$\begin{tikzcd}
& \{\mathbf{CD}\} \arrow[dr, dash] & & \{\mathbf{CE}\} \arrow[dl, dash] \\
\{\mathbf{AC}, \mathbf{ABC}\} \arrow[drr, dash] & & \{\mathbf{C}, \mathbf{BC}\} \arrow[d, dash] & \\
& & \{\mathbf{AB}\} \arrow[dl, dash] \arrow[dr, dash] & \\
& \{\mathbf{A}\} & & \{\mathbf{B}\}
\end{tikzcd} \tag{57}$$

8.2. **Solutions and paths.** We call a subset $A \subset \mathbb{Z}$ *left bounded* (respectively *right bounded*) if it has a minimum (respectively maximum). Otherwise we call it *left unbounded* or *left infinite* (respectively *right unbounded* or *right infinite*). We call $A \subset \mathbb{Z}$ *bounded* if A has both a minimum and a maximum. We call $A \subset \mathbb{Z}$ a $\mathbb{Z}$-interval if $A = \mathbb{Z} \cap I$ where I is an interval in $\mathbb{R}$.

A *solution* of a multivector field $\mathcal{V}$ in $A \subset X$ is a partial map $\varrho : \mathbb{Z} \nrightarrow A$ whose *domain*, denoted by $\operatorname{dom} \varrho$, is a $\mathbb{Z}$-interval and for any $i, i+1 \in \operatorname{dom} \varrho$ the inclusion $\varrho(i+1) \in \Pi_{\mathcal{V}}(\varrho(i))$ holds. The solution *passes* through $x \in X$ if $0 \in \operatorname{dom} \varrho$ and $x = \varrho(0)$ for some $i \in \operatorname{dom} \varrho$. The solution ϱ is *full* if

$\operatorname{dom}\varrho = \mathbb{Z}$. It is a *backward solution* if $\operatorname{dom}\varrho$ is left infinite. It is a *forward solution* if $\operatorname{dom}\varrho$ is right infinite. It is a *partial solution* or simply a *path* if $\operatorname{dom}\varrho$ is bounded. We refer to the cardinality of the domain of a path as the *length* of the path. If the maximum of $\operatorname{dom}\varrho$ exists, we call the value of ϱ at this maximum the *right endpoint* of ϱ. If the minimum of $\operatorname{dom}\varrho$ exists, we call the value of ϱ at this minimum the *left endpoint* of ϱ. We denote the left and right endpoints of ϱ by $\varrho^{\sqsubset}$ and $\varrho^{\sqsupset}$, respectively. Given a full solution ϱ through $x \in X$, we denote by $\varrho^+ := \varrho_{|\mathbb{Z}_0^+}$ the *forward solution* through x, and by $\varrho^- := \varrho_{|\mathbb{Z}_0^-}$ the *backward solution* through x.

By a *shift* of a solution ϱ we mean the composition $\varrho \circ \tau_n$, where $\tau_n : \mathbb{Z} \ni m \mapsto m + n \in \mathbb{Z}$ is the translation map. Given two solutions φ and ψ such that $\psi^{\sqsubset}$ and $\varphi^{\sqsupset}$ exist and $\psi^{\sqsubset} \in \Pi_{\mathcal{V}}(\varphi^{\sqsupset})$, there is a unique shift τ_n such that $\varphi \cup \psi \circ \tau_n$ is a solution. We call this union of paths the *concatenation* of φ and ψ and denote it $\varphi \cdot \psi$. We also identify each $x \in X$ with the path of length one whose image is $\{x\}$. The following proposition is straightforward.

Proposition 8.3. *Let $x, y \in X$. Then $y \in \Pi_{\mathcal{V}}(x)$ if and only if there is an arrow from $[x]$ to $[y]$ in the $\mathcal{V}$-digraph. Consequently, the multivector field $\mathcal{V}$ admits a path from x to y if and only if there is a path from $[x]$ to $[y]$ in the $\mathcal{V}$-digraph.* □

8.3. **Isolating neighborhoods and isolated invariant sets.** A full solution $\varrho : \mathbb{Z} \to X$ is called *left-essential* (respectively *right-essential*), if for every regular $x \in \operatorname{im}\varrho$ the set $\{\, t \in \mathbb{Z} \mid \varrho(t) \notin [x]_{\mathcal{V}} \,\}$ is *left-infinite* (respectively *right-infinite*). We say that ϱ is *essential* if it is both left- and right-essential. We say that $S \subset X$ is *$\mathcal{V}$-invariant*, or briefly *invariant*, if for every $x \in S$ there exists an essential solution through x in S. A closed set $N \subset X$ is an *isolating set* for a $\mathcal{V}$-invariant subset $S \subset N$ if $\Pi_{\mathcal{V}}(S) \subset N$ and any path in N with endpoints in S is a path in S. We say that S is an *isolated invariant set* if S admits an isolating set.

8.4. **Recurrent solutions and gradient-like multivector fields.** An essential solution $\varrho : \mathbb{Z} \to X$ is *recurrent* if for every $x \in \operatorname{im}\varrho$ the set $\varrho^{-1}(x)$ is both left- and right-infinite. A multivector field $\mathcal{V}$ is called *gradient-like* if for every recurrent solution $\varrho : \mathbb{Z} \to X$ there exists a multivector $V \in \mathcal{V}$ such that $\operatorname{im}\varrho \subset V$. A multivector field $\mathcal{V}$ is called *gradient* if the only recurrent solutions are constant solutions with its value in a singleton of $\mathcal{V}$. We would like to point out that this rules out multivectors of size at least two which have the index of a hyperbolic equilibrium. Moreover, recall that a singleton in $\mathcal{V}$ is always critical, see Proposition 7.6. We then have the following proposition.

Proposition 8.4. *Let $\mathcal{V}$ be a combinatorial multivector field on a Lefschetz complex X. The following properties are pairwise equivalent.*

(i) *$\mathcal{V}$ is gradient-like.*
(ii) *After discarding self-loops, the $\mathcal{V}$-digraph is acyclic.*

(iii) *$\mathcal{V}$ is an acyclic partition in X.*

Proof: The equivalence (i) $\Leftrightarrow$ (ii) follows from Proposition 8.3, and the equivalence (ii) $\Leftrightarrow$ (iii) follows from Proposition 8.1. $\square$

8.5. **Morse decompositions and connection matrices.** We define the *backward and forward ultimate image* of a full solution $\varrho : \mathbb{Z} \to X$ respectively as the sets

$$\begin{aligned} \operatorname{uim}^-(\varrho) &:= \bigcap_{t \in \mathbb{Z}^-} \varrho((-\infty, t]), \\ \operatorname{uim}^+(\varrho) &:= \bigcap_{t \in \mathbb{Z}^+} \varrho([t, \infty)). \end{aligned}$$

The following proposition is straightforward

Proposition 8.5. *If $\varrho : \mathbb{Z} \to X$ is a periodic solution, then its backward and forward ultimate images satisfy* $\operatorname{uim}^-(\varrho) = \operatorname{im} \varrho = \operatorname{uim}^+(\varrho)$. $\square$

By a *Morse decomposition of $\mathcal{V}$* we mean a collection $\mathcal{M}$ of mutually disjoint isolated invariant sets of $\mathcal{V}$, called *Morse sets*, together with a partial order $\leq$ on $\mathcal{M}$ such that for every essential solution $\varrho : \mathbb{Z} \to X$ one either has $\operatorname{im} \varrho \subset M$ for a Morse set $M \in \mathcal{M}$, or there exist Morse sets $M, M' \in \mathcal{M}$ which satisfy $M < M'$ and for which $\operatorname{uim}^-(\varrho) \subset M'$ and $\operatorname{uim}^+(\varrho) \subset M$.

Let $\mathcal{M}$ be a Morse decomposition of $\mathcal{V}$. Then the family

$$\mathcal{E}_{\mathcal{M}} := \mathcal{M} \cup \{\, V \in \mathcal{V} \mid V \cap \bigcup \mathcal{M} = \varnothing \,\}$$

is clearly a partition of X. We call it the *partition induced by the Morse decomposition* $\mathcal{M}$. Each $E \in \mathcal{E}_{\mathcal{M}}$ either is a Morse set, or it is a multivector not contained in any of the Morse sets. For $E, E' \in \mathcal{E}_{\mathcal{M}}$ we write $E \preccurlyeq E'$ if and only if there is a path σ such that $\sigma^{\sqsubset} \in E'$ and $\sigma^{\sqsupset} \in E$.

Proposition 8.6. *The relation $\preccurlyeq$ is a partial order in $\mathcal{E}_{\mathcal{M}}$.*

Proof: Clearly, the relation is reflexive and transitive. To see that it is antisymmetric, take $E, E' \in \mathcal{E}_{\mathcal{M}}$ such that $E \preccurlyeq E'$ and $E' \preccurlyeq E$. We will prove that the assumption $E \neq E'$ leads to a contradiction. Let σ be a path such that $\sigma^{\sqsubset} \in E'$ and $\sigma^{\sqsupset} \in E$. In addition, let τ denote a path which satisfies both $\tau^{\sqsubset} \in E$ and $\tau^{\sqsupset} \in E'$.

Consider first the case when E and E' are multivectors disjoint from Morse sets. Then $[\sigma^{\sqsubset}] = E' = [\tau^{\sqsupset}]$ and $[\sigma^{\sqsupset}] = E = [\tau^{\sqsubset}]$. It follows that the paths σ and τ can be concatenated to a full, periodic solution ϱ. Since we assumed that $E \neq E'$, this solution ϱ is essential. Consequently, $\operatorname{uim}^-(\varrho)$ and $\operatorname{uim}^+(\varrho)$ must be contained in Morse sets. However, it follows from Proposition 8.5 that $\operatorname{uim}^-(\varrho) = \operatorname{im} \varrho = \operatorname{uim}^+(\varrho)$, and therefore the point $\sigma^{\sqsubset}$ is contained in both E' and a Morse set, which contradicts our assumption that both E and E' are multivectors disjoint from Morse sets. Thus, we have $E = E'$ in this case.

Consider now the case when both E and E' are Morse sets. In particular, this implies that they are invariant. Let ϱ be an essential solution through $\sigma^{\sqsubset}$ in E', and let π be an essential solution through $\sigma^{\sqsupset}$ in E. Then $\bar{\sigma} := \varrho^{-}\cdot\sigma\cdot\pi^{+}$ is an essential solution with $\operatorname{uim}^{-}(\bar{\sigma}) \subset E'$ and $\operatorname{uim}^{+}(\bar{\sigma}) \subset E$, and therefore it follows from the definition of Morse decomposition that $E' > E$. Similarly, we can extend τ to an essential solution $\bar{\tau}$ which satisfies both $\operatorname{uim}^{-}(\bar{\tau}) \subset E$ and $\operatorname{uim}^{+}(\bar{\tau}) \subset E'$. Hence, one obtains $E > E'$, another contradiction.

Finally, consider the case when one of the sets E, E' is a Morse set and the other is a multivector disjoint from Morse sets. Without loss of generality we may assume that E is a Morse set and E' is a multivector disjoint from Morse sets. Then $[\sigma^{\sqsubset}] = E' = [\tau^{\sqsupset}]$. Let π be an essential solution through $\sigma^{\sqsupset}$ in the Morse set E, and let ϱ be an essential solution through $\tau^{\sqsubset}$ in E. Then the concatenation $\xi := \varrho^{-} \cdot \tau \cdot \sigma \cdot \pi^{+}$ is an essential solution which satisfies both $\operatorname{uim}^{-}(\xi) \subset E$ and $\operatorname{uim}^{+}(\xi) \subset E$. Thus, it follows from the definition of Morse decomposition that $\operatorname{im}\xi \subset E$, a contradiction. □

Proposition 8.7. *The relations $\preccurlyeq$ and $\leq_{\mathcal{E}_{\mathcal{M}}}$ coincide. In particular, the family $\mathcal{E}_{\mathcal{M}}$ is an acyclic partition of X.*

Proof: We need to prove that for all $E, E' \in \mathcal{E}_{\mathcal{M}}$

$$E \leq_{\mathcal{E}_{\mathcal{M}}} E' \quad \Leftrightarrow \quad E \preccurlyeq E' \tag{58}$$

First, we will prove that

$$E \cap \operatorname{cl} E' \neq \varnothing \quad \Rightarrow \quad E \preccurlyeq E'. \tag{59}$$

Let $x \in E \cap \operatorname{cl} E'$. Then $x \in \operatorname{cl} y$ for a $y \in E'$. Thus, $\sigma := y \cdot x$ is a path. Moreover, $\sigma^{\sqsubset} = y \in E'$ and $\sigma^{\sqsupset} = x \in E$, proving (59). Since, by Proposition 8.6, the relation $\preccurlyeq$ is transitive, it follows from property (59) that the left-hand side of (58) implies the right-hand side of (58).

For the reverse implication assume that $E \preccurlyeq E'$. Let $\varrho = x_0 \cdot x_1 \cdot \ldots \cdot x_n$ be a path from E' to E. Let $V_i := [x_i]$. Then $V_0 \subset E'$ and $V_n \subset E$. Moreover, since $x_{i+1} \in \operatorname{cl} x_i \cup [x_i]$, we see that $V_{i+1} \cap \operatorname{cl} V_i \neq \varnothing$ or $V_{i+1} = V_i$. It follows that $V_{i+1} \preceq_{\mathcal{E}_{\mathcal{M}}} V_i$, and therefore $E \leq_{\mathcal{E}_{\mathcal{M}}} E'$. Thus, (58) is proved. Hence, it follows from Proposition 8.6 that $\leq_{\mathcal{E}_{\mathcal{M}}}$ is a partial order on $\mathcal{E}_{\mathcal{M}}$, and this finally implies that $\mathcal{E}_{\mathcal{M}}$ is an acyclic partition. □

Proposition 8.7 lets us define the Conley complex and connection matrices of Morse decompositions.

Definition 8.8. The *Conley complex* and the associated *connection matrices* of a Morse decomposition $\mathcal{M}$ of a combinatorial multivector field on a Lefschetz complex are defined as the Conley complex and the associated connection matrices of the acyclic partition $\mathcal{E}_{\mathcal{M}}$, i.e., the Conley complex and connection matrices of the poset filtered chain complex $(\mathcal{E}_{\mathcal{M}}, C(X), \partial^{\kappa})$.

Example 8.9. All three combinatorial multivector fields $\mathcal{V}_0$, $\mathcal{V}_1$, $\mathcal{V}_2$ in Figure 5 have a common Morse decomposition given by the sets

$$\mathcal{M}_0 = \mathcal{M}_1 = \mathcal{M}_2 = \{\{\mathbf{A}\}, \{\mathbf{B}\}, \{\mathbf{AB}\}, \{\mathbf{CD}\}, \{\mathbf{CE}\}\},$$

which consist of singletons. However, the acyclic partition $\mathcal{E}_{\mathcal{M}_i}$ is different for each index $i \in \{0, 1, 2\}$. In these simple examples, $\mathcal{E}_{\mathcal{M}_i}$ coincides with $\mathcal{V}_i$. The filtered chain complex $(\mathcal{E}_{\mathcal{M}_0}, C(X), \partial^\kappa)$ coincides with the filtered chain complex considered in Example 5.9. Therefore, the connection matrices given by (27) and (31) are the connection matrices of $\mathcal{M}_0$. We know that these matrices are not equivalent.

9. Connection matrices for Forman's gradient vector fields

Recall that a combinatorial multivector W is a *combinatorial vector* if $\operatorname{card} W \leq 2$. A combinatorial multivector field whose multivectors are just vectors is called a *combinatorial vector field*, a concept introduced by R. Forman [6, 5]. In the case of a combinatorial vector field the critical vectors are precisely the singletons and the regular vectors are precisely the doubletons, i.e., vectors of cardinality two. Therefore, every gradient-like combinatorial vector field is a gradient combinatorial vector field.

In this section we assume that $\mathcal{V}$ is a gradient combinatorial vector field on a Lefschetz complex X. Let $\mathcal{C} \subset \mathcal{V}$ be the collection of critical vectors in $\mathcal{V}$. Since we assume that $\mathcal{V}$ is a combinatorial vector field, it follows from Proposition 7.6 that $\mathcal{C}$ is exactly the collection of singletons in $\mathcal{V}$. The following proposition is straightforward.

Proposition 9.1. *Assume $\mathcal{V}$ is a gradient combinatorial vector field. The collection $\mathcal{C}$ is a Morse decomposition of $\mathcal{V}$ and the partition of X induced by this Morse decomposition is $\mathcal{E}_{\mathcal{C}} = \mathcal{V}$.* □

It follows that the Conley complex of the Morse decomposition $\mathcal{C}$ of X is the Conley complex of the filtered chain complex $(\mathcal{V}, C(X), \partial^\kappa)$. The aim of this section is to prove that the Morse decomposition $\mathcal{C}$ has precisely one connection matrix, and that this connection matrix coincides with the matrix of the boundary operator of the associated Morse complex, see also [6, Section 7]. In order to make this statement precise, we first need to recall some concepts.

Unlike a general multivector, a combinatorial vector $W \in \mathcal{V}$ contains a unique minimal element and a unique maximal element in W with respect to $\leq_\kappa$. We denote them by W^- and W^+, respectively, and we extend the notation to cells by writing $x^- := [x]^-$ and $x^+ := [x]^+$. Note that a combinatorial vector is given by $W = \{W^-, W^+\}$, and W is critical if and only if we have $W^+ = W^-$, which in turn is equivalent to assuming that W is a singleton. As a consequence of Proposition 7.4 one obtains for a vector W the inequality

$$0 \leq \dim W^+ - \dim W^- \leq 1, \tag{60}$$

and for all $x \in W$ one has

$$\dim W^- \leq \dim x \leq \dim W^+. \tag{61}$$

We say that a cell x is a *tail* if $x = x^- \neq x^+$ and a *head* if $x = x^+ \neq x^-$. Clearly, if a cell is neither a tail nor a head, then it is critical. We denote the subsets of critical cells, tails and heads respectively by $X^c(\mathcal{V})$, $X^-(\mathcal{V})$, and $X^+(\mathcal{V})$, and we shorten this notation to X^c, X^-, and X^+ whenever $\mathcal{V}$ is clear from the context. Note that the collection $\mathcal{C}$ of critical vectors of $\mathcal{V}$ is precisely $\{\, \{x\} \mid x \in X^c \,\}$.

For a combinatorial vector $V \in \mathcal{V}$ we set $\dim V := \dim V^+$. Moreover, recall that $\mathcal{V}$, as a gradient vector field, is an acyclic partition of X according to Proposition 8.4. In particular, $\mathcal{V}$ is a poset with the partial order $\leq_{\mathcal{V}}$. We have the following proposition.

Proposition 9.2. *Assume that $\mathcal{V}$ is a gradient vector field on a Lefschetz complex X. Then the following hold:*

- (i) *The map* $\dim : (\mathcal{V}, \leq_{\mathcal{V}}) \to (\mathbb{N}_0, \leq)$ *is order preserving.*
- (ii) *If $V <_{\mathcal{V}} W$ and V is critical, then* $\dim V < \dim W$*.*
- (iii) *We have $\mathcal{V}_\star = \mathcal{C}$, that is, the set of $V \in \mathcal{V}$ such that $C(V)$ is homotopically essential coincides with the set of critical vectors $\mathcal{C}$.*

Proof: Since, by definition, the relation $\leq_{\mathcal{V}}$ is the transitive closure of the relation $\preceq_{\mathcal{V}}$, to prove (i) it suffices to verify that $V \cap \operatorname{cl} W \neq \varnothing$ for $V, W \in \mathcal{V}$ implies $\dim V \leq \dim W$. Thus, assume that $V \cap \operatorname{cl} W \neq \varnothing$. Let $x \in V \cap \operatorname{cl} W$. If $x \in W$, then $V \cap W \neq \varnothing$ which implies $V = W$, because $\mathcal{V}$ is a partition. In particular, this gives $\dim V = \dim W$. Thus, consider now case $x \notin W$. Since $W^- \in \operatorname{cl} W^+$, we have $\operatorname{cl} W = \operatorname{cl}\{W^-, W^+\} = \operatorname{cl} W^- \cup \operatorname{cl} W^+ = \operatorname{cl} W^+$. It follows that $x \in \operatorname{cl} W^+ \setminus \{W^+\}$, and therefore $\dim x < \dim W^+$. Since we assumed $x \in V$, we get from (60) and (61) that $\dim x \geq \dim V^- \geq \dim V^+ - 1 = \dim V - 1$. Thus, $\dim V \leq \dim x + 1 \leq \dim W^+ = \dim W$.

To see (ii), without loss of generality, we may assume that $V \prec_{\mathcal{V}} W$, that is, $V \neq W$ and $V \cap \operatorname{cl} W \neq \varnothing$. Observe that since V is critical, both $V = \{x\}$ and $\dim V = \dim x$ are satisfied. Since $x \notin W$ and $x \in \operatorname{cl} W = \operatorname{cl} W^+$, we have $\dim x < \dim W^+$. Therefore, $\dim V = \dim x < \dim W^+ = \dim W$.

To see (iii) observe that, by Proposition 7.6, the Lefschetz homology of V is zero if and only if V is not critical. Therefore, property (iii) follows immediately from Theorem 5.6(iv). $\square$

Consider now a regular Lefschetz complex X. With a given combinatorial vector field $\mathcal{V}$ we can associate the degree $+1$ map $\Gamma_{\mathcal{V}} : C(X) \to C(X)$, which for a generator $x \in X$ is defined by

$$\Gamma_{\mathcal{V}} x := \begin{cases} 0 & \text{if } x^- = x^+, \\ -\kappa(x^+, x^-)^{-1} \langle x, x^- \rangle x^+ & \text{otherwise.} \end{cases} \tag{62}$$

In the sequel, we drop the subscript in $\Gamma_{\mathcal{V}}$ whenever $\mathcal{V}$ is clear from the context. The following proposition is straightforward.

Proposition 9.3. *If X is a regular Lefschetz complex and $\mathcal{V}$ is a combinatorial vector field on X, then for every $c \in C(X)$ we have $|\Gamma c| \subset X^+$.*

Proposition 9.4. *Suppose that X is a regular Lefschetz complex and $\mathcal{V}$ is a combinatorial vector field on X. Then for all $x, u \in X$ we have both the implication*

$$\langle u, \Gamma\partial x\rangle \neq 0 \quad \Rightarrow \quad u \in X^+ \text{ and } \langle u, \Gamma\partial x\rangle = -\kappa(x, u^-)\kappa(u^+, u^-)^{-1} \tag{63}$$

and

$$\langle u, \partial\Gamma x\rangle \neq 0 \quad \Rightarrow \quad x \in X^- \text{ and } \langle u, \partial\Gamma x\rangle = -\kappa(x^+, x^-)^{-1}\langle u, \partial x^+\rangle. \tag{64}$$

Proof: The definition of the boundary operator of a Lefschetz complex given in (49) implies

$$\begin{aligned}\Gamma\partial x &= \Gamma\left(\sum_{y\in X}\kappa(x,y)y\right) = \sum_{y\in X}\kappa(x,y)\Gamma y\\ &= -\sum_{y\in X,\ \Gamma y\neq 0}\kappa(x,y)\kappa(y^+,y^-)^{-1}\langle y, y^-\rangle y^+.\end{aligned}$$

Hence, we have

$$\langle u, \Gamma\partial x\rangle = -\sum_{y\in X,\ \Gamma y\neq 0}\kappa(x,y)\kappa(y^+,y^-)^{-1}\langle y, y^-\rangle\langle u, y^+\rangle.$$

Thus, if $\langle u, \Gamma\partial x\rangle$ is non-zero, then there is a $y \in X$ such that $y^- \neq y^+$ and that all factors in $\kappa(x,y)\langle y, y^-\rangle\langle u, y^+\rangle$ are non-zero. In particular, the inequality $\langle y, y^-\rangle \neq 0$ implies $y = y^-$. Hence, $y \in X^-$ and from $\langle u, y^+\rangle \neq 0$ we get $u = y^+ \in X^+$ and $y = y^- = [y^+]^- = [u]^- = u^-$. It follows that such a y is unique and $\langle u, \Gamma\partial x\rangle = -\kappa(x,y)\kappa(y^+,y^-)^{-1} = -\kappa(x,u^-)\kappa(u^+,u^-)^{-1}$. This proves (63).

To see (64) observe that if $\langle u, \partial\Gamma x\rangle \neq 0$, then $\Gamma x \neq 0$. Hence, $x^- \neq x^+$ and one further obtains

$$\langle u, \partial\Gamma x\rangle = \langle u, -\kappa(x^+,x^-)^{-1}\langle x, x^-\rangle\partial x^+\rangle = -\kappa(x^+,x^-)^{-1}\langle x, x^-\rangle\langle u, \partial x^+\rangle.$$

Therefore, if $\langle u, \partial\Gamma x\rangle \neq 0$, then $\langle x, x^-\rangle \neq 0 \neq \langle u, \partial x^+\rangle$. In particular, we get $x = x^-$, which in turn means that $x \in X^-$ and

$$\langle u, \partial\Gamma x\rangle = -\kappa(x^+,x^-)^{-1}\langle u, \partial x^+\rangle.$$

This identity finally proves (64). $\square$

Given a combinatorial vector field $\mathcal{V}$, we define the associated *combinatorial flow* as in [6, Theorem 6.4] by letting $\Phi := \Phi_{\mathcal{V}} : C(X) \to C(X)$ act on a generator $x \in X$ through the formula

$$\Phi_{\mathcal{V}}x := x + \partial\Gamma_{\mathcal{V}}x + \Gamma_{\mathcal{V}}\partial x. \tag{65}$$

In the sequel, whenever $\mathcal{V}$ is clear from the context, we drop the subscript in $\Phi_{\mathcal{V}}$. Note that Φ is a degree zero module homomorphism which satisfies the identity $\Phi\partial = \partial + \partial\Gamma\partial = \partial\Phi$. Hence, Φ is a chain map. We have the following proposition.

Proposition 9.5. *Suppose that X is a regular Lefschetz complex and $\mathcal{V}$ is a combinatorial vector field on X. Then for all $x \in X$ we have*

$$\langle x, \Phi x\rangle = \begin{cases} 1 & \text{if } x \in X^c, \\ 0 & \text{otherwise.} \end{cases}$$

Proof: We have $\langle x, \Phi x\rangle = \langle x, x\rangle + \langle x, \partial\Gamma x\rangle + \langle x, \Gamma\partial x\rangle$. In the case $x \in X^c$ we get from Proposition 9.4 that $\langle x, \partial\Gamma x\rangle = 0$ and $\langle x, \Gamma\partial x\rangle = 0$, which implies $\langle x, \Phi x\rangle = \langle x, x\rangle = 1$.

Consider now the case $x \in X^-$. Then one has $x = x^- \notin X^+$ and we get from (63) that $\langle x, \Gamma\partial x\rangle = 0$. Therefore, the definition (62) yields

$$\begin{aligned}\langle x, \Phi x\rangle &= \langle x, x\rangle + \langle x, \partial\Gamma x\rangle = \\ &\quad 1 - \kappa(x^+, x^-)^{-1}\langle x, x^-\rangle\langle x, \partial x^+\rangle = 1 - \kappa(x^+, x^-)^{-1}\kappa(x^+, x^-) = 0.\end{aligned}$$

Finally, we consider the remaining case $x \in X^+$. Then $x \notin X^-$, and (64) shows that $\langle x, \partial\Gamma x\rangle = 0$. Therefore, since $\partial x = \sum_{y \in X}\kappa(x, y)y$, we get

$$\begin{aligned}\langle x, \Phi x\rangle = \langle x, x\rangle + \langle x, \Gamma\partial x\rangle &= 1 + \langle x, \sum_{y\in X}\kappa(x, y)\Gamma y\rangle \\ &= 1 - \langle x, \sum_{y\in X,\ \Gamma y\neq 0}\kappa(x, y)\kappa(y^+, y^-)^{-1}\langle y, y^-\rangle y^+\rangle \\ &= 1 - \sum_{y\in X,\ \Gamma y\neq 0}\kappa(x, y)\kappa(y^+, y^-)^{-1}\langle y, y^-\rangle\langle x, y^+\rangle \\ &= 1 - \kappa(x^+, x^-)\kappa(x^+, x^-)^{-1} = 0,\end{aligned}$$

because the only non-zero term in the last sum occurs for the point $y \in X$ which satisfies both $y^+ = x = x^+$ and $y = y^- = x^-$. $\square$

Proposition 9.6. *Suppose that X is a regular Lefschetz complex and $\mathcal{V}$ is a combinatorial vector field on X. Then we have*

$$\Phi(C(X^+)) \subset C(X^+).$$

Proof: Since Φ is linear, it suffices to show that the inclusion $x \in X^+$ implies $|\Phi x| \subset X^+$. Thus, take $x \in X^+$ and a $u \in |\Phi x|$. Then $\langle u, \Phi x\rangle \neq 0$. From (64) we get $\langle u, \partial\Gamma x\rangle = 0$. Therefore, $0 \neq \langle u, \Phi x\rangle = \langle u, x\rangle + \langle u, \Gamma\partial x\rangle$, which implies $\langle u, x\rangle \neq 0$ or $\langle u, \Gamma\partial x\rangle \neq 0$. If $\langle u, x\rangle \neq 0$, then $u = x \in X^+$. If the inequality $\langle u, \Gamma\partial x\rangle \neq 0$ holds, then we get from (63) that $u \in X^+$. Hence, $|\Phi x| \subset X^+$. $\square$

Proposition 9.7. *Suppose that X is a regular Lefschetz complex and $\mathcal{V}$ is a combinatorial vector field on X. Assume further that $\mathcal{A} \in \mathrm{Down}(\mathcal{V}, \leq_{\mathcal{V}})$. Then $A := |\mathcal{A}|$ is a $\mathcal{V}$-compatible and closed subset of X, and we have the inclusion $\Phi(C(A)) \subset C(A)$. Thus, $\Phi_{|C(A)} : (C(A), \partial^\kappa_{|C(A)}) \to (C(A), \partial^\kappa_{|C(A)})$ is a well-defined chain map obtained through restriction.*

Proof: Clearly, A is $\mathcal{V}$-compatible, and it follows from Proposition 7.9(i) that A is closed. Take an $x \in A$. We will prove that $\Phi x \in C(A)$. Since A is closed, we have $|\partial x| \subset A$. Since A is $\mathcal{V}$-compatible, we have $|\Gamma_{\mathcal{V}} x| \subset A$. Thus, again by closedness and $\mathcal{V}$-compatibility of A, we get both $|\partial \Gamma_{\mathcal{V}} x| \subset A$ and $|\Gamma_{\mathcal{V}} \partial x| \subset A$. Therefore,

$$|\Phi x| = |x + \partial \Gamma_{\mathcal{V}} x + \Gamma_{\mathcal{V}} \partial x| \subset |x| \cup |\partial \Gamma_{\mathcal{V}} x| \cup |\Gamma_{\mathcal{V}} \partial x| \subset A.$$

This finally implies $\Phi x \in C(A)$ and completes the proof. □

Proposition 9.8. *Suppose that X is a regular Lefschetz complex and $\mathcal{V}$ is a combinatorial vector field on X. Assume further that $x, y \in X$. Then the following hold:*

(i) *If $y \in |\Phi x|$, then $[y] \leq_{\mathcal{V}} [x]$.*
(ii) *If $y \in |\Phi x|$ and $[y] = [x]$, then $y = x$.*
(iii) *If $x \in |\Phi x|$, then $\langle x, \Phi x \rangle = 1$.*

Proof: In order to see (i), observe that the assumption $y \in |\Phi x|$ implies the inequality $0 \neq \langle y, \Phi x \rangle = \langle y, x \rangle + \langle y, \partial \Gamma x \rangle + \langle y, \Gamma \partial x \rangle$. Thus, either we have $\langle y, x \rangle \neq 0$, or $\langle y, \partial \Gamma x \rangle \neq 0$, or $\langle y, \Gamma \partial x \rangle \neq 0$. In the first case one immediately obtains $x = y$ and $[x] = [y]$. Consider now the second case $\langle y, \partial \Gamma x \rangle \neq 0$. This inequality yields both $\Gamma x \neq 0$ and $y \in |\partial \Gamma x|$. Hence, we have $x = x^-$ and $\Gamma x = \lambda x^+$ with $\lambda = -\kappa(x^+, x^-)^{-1} \neq 0$. This gives $|\partial \Gamma x| = |\partial x^+|$, as well as $y \in |\partial x^+| \subset \operatorname{cl} x^+$. It follows that $y \in [y] \cap \operatorname{cl}[x^+] = [y] \cap \operatorname{cl}[x^-]$, and therefore $[y] \leq_{\mathcal{V}} [x]$. Finally, we consider the third inequality $\langle y, \Gamma \partial x \rangle \neq 0$. Since $\partial x = \sum_{u \in X} \kappa(x, u) u$, we have

$$0 \neq \langle y, \Gamma \partial x \rangle = \sum_{u \in X} \kappa(x, u) \langle y, \Gamma u \rangle,$$

which implies $\langle y, \Gamma u \rangle \neq 0$ for some $u \in |\partial x|$. But this implies that $u = u^-$ and $\Gamma u = \lambda u^+$ with $\lambda = -\kappa(u^+, u^-)^{-1} \neq 0$. Hence, the identities $y = u^+$ and $[y] = [u^+] = [u^-]$ are satisfied, as well as the inclusion $u^- \in |\partial x| \subset \operatorname{cl} x$. Thus, $[y] \cap \operatorname{cl}[x] \neq \varnothing$, which implies $[y] \leq_{\mathcal{V}} [x]$. This finally completes the proof of (i).

In order to prove (ii), we observe that $y \in |\Phi x|$ implies $\dim x = \dim y$. Since $\mathcal{V}$ is a combinatorial vector field, we further have both $[x] = \{x^-, x^+\}$ and $[y] = \{y^-, y^+\}$. Thus, $[x] = [y]$ and $y \in |\Phi x|$ imply the identity $x = y$, which in turn proves (ii).

Finally consider (iii). If $x \in |\Phi x|$, then $\langle x, \Phi x \rangle \neq 0$. Thus, we get from Proposition 9.5 that $\langle x, \Phi x \rangle = 1$. □

We will use the fact that Φ stabilizes under iteration, that is, $\Phi^n = \Phi^{n+1}$ for n sufficiently large. This is established in [6, Theorem 7.2]. One can readily verify that while Forman only considers $\mathbb{Z}$ coefficients, his arguments remain valid for the case of general ring coefficients. Let $\Phi^\infty := \Phi^n = \Phi^{n+1}$ denote this stabilized value.

Proposition 9.9. *Assume that $\mathcal{V}$ is a gradient vector field on a regular Lefschetz complex X. If $y \in |\Phi^\infty x|$, then $[y] \leq_{\mathcal{V}} [x]$.*

Proof: It suffices to prove that $y \in |\Phi^n x|$ implies $[y] \leq_{\mathcal{V}} [x]$ for every positive integer $n \in \mathbb{N}$. We proceed by induction on n. For $n = 1$ the conclusion is the statement of Proposition 9.8. Thus, fix a $k \in \mathbb{N}$ and assume that the conclusion holds for all $n \in \{1, 2, \dots k\}$. Let $y \in |\Phi^{k+1}x|$ and let $c := \Phi^k x$. We have $c = \sum_{w \in |c|} \langle c, w \rangle w$ and $\Phi c = \sum_{w \in |c|} \langle c, w \rangle \Phi w$. Since

$$y \in |\Phi^{k+1}x| = |\Phi c| \subset \bigcup_{w \in |c|} |\Phi w|,$$

we get $y \in |\Phi w|$ for some $w \in |c| = |\Phi^k x|$. Thus, by our induction assumption, we obtain $[y] \leq_{\mathcal{V}} [w] \leq_{\mathcal{V}} [x]$. □

We have the following proposition.

Proposition 9.10. *Assume that $\mathcal{V}$ is a gradient vector field on a regular Lefschetz complex X. For every $x \in X^c$ we have*

$$\Phi^\infty x = x + r_x \tag{66}$$

where r_x is a chain satisfying $|r_x| \subset X^+ \cap |\, [x]^{<_\mathcal{V}} \,|$. In particular, $\Phi^\infty_{|X^c}$ is injective.

Proof: Let $x \in X^c$. Since $\Phi^\infty = \Phi^n$ for a large $n \in \mathbb{N}$, it suffices to verify that for every positive integer $n \in \mathbb{N}$

$$\Phi^n x = x + r_n \tag{67}$$

where r_n is a chain satisfying

$$|r_n| \subset X^+ \cap |\, [x]^{<_\mathcal{V}} \,|. \tag{68}$$

We proceed by induction on n. Assume first that $n = 1$ and set $r_1 := \Gamma \partial x$. From Proposition 9.3 we see that $|r_1| \subset X^+$ and, since $x \in X^c$, one obtains further $\langle x, r_1 \rangle = 0$. It follows from $x \in X^c$ that $\Gamma x = 0$. Therefore, the identity $\Phi x = x + \Gamma \partial x = x + r_1$ holds. To show that $|r_1| \subset |\, [x]^{<_\mathcal{V}} \,|$, take an arbitrary $y \in |r_1|$. Then $\langle y, r_1 \rangle \neq 0$, and since $\langle x, r_1 \rangle = 0$, we get $x \neq y$, that is, $\langle y, x \rangle = 0$. It follows that $\langle y, \Phi x \rangle = \langle y, x \rangle + \langle y, r_1 \rangle = \langle y, r_1 \rangle \neq 0$. Hence, $y \in |\Phi x|$. Thus, Proposition 9.8(i) gives $[y] \leq_{\mathcal{V}} [x]$ and $[y] \in [x]^{\leq_\mathcal{V}}$. Since $y \neq x$, and in view of the fact that Φ is a degree 0 map, one further has $\dim y = \dim x$, and we also get $[y] \neq [x]$ and $[y] \in [x]^{<_\mathcal{V}}$. This in turn yields $y \in |\, [x]^{<_\mathcal{V}} \,|$. Therefore, $|r_1| \subset |\, [x]^{<_\mathcal{V}} \,|$ and the proof of (67) for $n = 1$ is complete.

Next, fix a $k \in \mathbb{N}$ and assume that (67) holds for all $n \leq k$ with r_n satisfying (68). We then have $\Phi^{k+1}x = \Phi(\Phi^k x) = \Phi x + \Phi r_k = x + r_1 + \Phi r_k$, and let $r_{k+1} := r_1 + \Phi r_k$. It follows from the induction assumption and Proposition 9.6 that $|r_{k+1}| \subset |r_1| \cup |\Phi r_k| \subset X^+$. Since $|r_1| \subset |\, [x]^{<_\mathcal{V}} \,|$, it suffices to prove that $|\Phi r_k| \subset |\, [x]^{<_\mathcal{V}} \,|$ in order to see that $|r_{k+1}| \subset |\, [x]^{<_\mathcal{V}} \,|$. For this, take an arbitrary element $y \in |r_k|$. Then the induction assumption gives $y \in |\, [x]^{<_\mathcal{V}} \,|$. Since $|\, [x]^{<_\mathcal{V}} \,|$ is $\mathcal{V}$-compatible, it follows that $[y] \in [x]^{<_\mathcal{V}}$.

Since $[x]^{<\nu}$ is a down set, we get $[y]^{\leq\nu} \subset [x]^{<\nu}$ and $|\,[y]^{\leq\nu}\,| \subset |\,[x]^{<\nu}\,|$. Thus, we proved that

$$y \in |r_k| \quad \Rightarrow \quad |\,[y]^{\leq\nu}\,| \subset |\,[x]^{<\nu}\,|. \tag{69}$$

We have $r_k = \sum_{y\in|r_k|}\langle r_k, y\rangle y$ and $\Phi r_k = \sum_{y\in|r_k|}\langle r_k, y\rangle \Phi y$. By Proposition 9.8(i) and property (69)

$$|\Phi r_k| \subset \bigcup_{y\in|r_k|} |\Phi y| \subset \bigcup_{y\in|r_k|} |\,[y]^{\leq\nu}| \subset |\,[x]^{<\nu}\,|.$$

Thus,

$$|r_{k+1}| \ \subset \ |r_1| \ \cup \ |\Phi r_k| \ \subset \ |\,[x]^{<\nu}\,| \ \cup \ |\,[x]^{<\nu}\,| \ = \ |\,[x]^{<\nu}\,|,$$

which completes the induction argument.

Finally, to prove that $\Phi^\infty_{|X^c}$ is injective we will show the more general statement that $\Phi^\infty_{|C(X^c)}$ is a monomorphism. For this, it suffices to verify that $\Phi^\infty c = 0$ implies $c = 0$ for $c \in C(X^c)$. Let $c \in C(X^c)$. Then

$$c = \sum_{x\in X^c} a_x x,$$

and by (66)

$$0 = \Phi^\infty c = \sum_{x\in X^c} a_x \Phi^\infty x = c + \sum_{x\in X^c} a_x r_x \in C(X^c) \oplus C(X^+).$$

Hence, it follows that $c = 0$. □

Consider the set

$$\operatorname{Fix}\Phi := \{\, c \in C(X) \mid \Phi c = c \,\}$$

consisting of chains fixed by Φ. Clearly, if $c \in \operatorname{Fix}\Phi$, then $\partial c \in \operatorname{Fix}\Phi$, because $\Phi\partial c = \partial\Phi c = \partial c$. It follows that $(\operatorname{Fix}\Phi, \partial_{|\operatorname{Fix}\Phi})$ is a chain subcomplex of $(C(X), \partial)$ and $\Phi^\infty : (C(X), \partial) \to (\operatorname{Fix}\Phi, \partial_{|\operatorname{Fix}\Phi})$ is a chain epimorphism. For this, note that $\Phi^\infty = \Phi^n$ for large n and that Φ is a chain map.

Recall that $X^c \subset X$ stands for the collection of critical cells of $\mathcal{V}$. For every $x \in X^c$ we define $\bar{x} := \Phi^\infty x \in \operatorname{Fix}\Phi$.

Proposition 9.11. *Assume that $\mathcal{V}$ is a gradient vector field on a regular Lefschetz complex X. The the following hold:*

(i) *The set*

$$\bar{X} := \{\, \bar{x} \mid x \in X^c \,\}$$

is a basis of $\operatorname{Fix}\Phi$.

(ii) *For every $x \in X^c$ we have*

$$\partial\bar{x} = \sum_{z\in X^c} a_{xz}\bar{z}, \tag{70}$$

for uniquely determined coefficients $a_{xz} \in R$.

(iii) *The pair $(\bar{X}, \bar{\kappa})$, where the $\mathbb{Z}$-gradation of $\bar{X}$ is induced by the map $\dim : \bar{X} \ni \bar{x} \mapsto \dim x \in \mathbb{Z}$ and $\bar{\kappa}(\bar{x}, \bar{z})$ is the coefficient a_{xz} in (70), is a Lefschetz complex and $C(\bar{X}) = \operatorname{Fix}\Phi$.*

Proof: In order to prove (i) consider the projection $\Pi : \operatorname{Fix}\Phi \to C(X^c)$ given for $c \in \operatorname{Fix}\Phi$ by $\Pi c := \sum_{x \in X^c} \langle c, x\rangle x$. It follows from the proof of [6, Theorem 8.2] that $\Phi^\infty \Pi = \mathrm{id}_{\operatorname{Fix}\Phi}$. Take a $c \in \operatorname{Fix}\Phi$. Then

$$c = \Phi^\infty(\Pi c) = \Phi^\infty(\sum_{x \in X^c} \langle c, x\rangle x) = \sum_{x \in X^c} \langle c, x\rangle \Phi^\infty x = \sum_{x \in X^c} \langle c, x\rangle \bar{x}$$

proving that $\bar{X}$ generates $\operatorname{Fix}\Phi$. To see that $\bar{X}$ is linearly independent, assume that

$$\sum_{x \in X^c} a_x \bar{x} = 0$$

for some coefficients $a_x \in R$. Then $0 = \sum_{x \in X^c} a_x \Phi^\infty x = \Phi^\infty(\sum_{x \in X^c} a_x x)$. Since $\sum_{x \in X^c} a_x x \in C(X^c)$, and $\Phi_{|C(X^c)} : C(X^c) \to \operatorname{Fix}\Phi$ is an isomorphism due to [6, Theorem 8.2], we conclude that $\sum_{x \in X^c} a_x x = 0$. This in turn implies $a_x = 0$ for all $x \in X^c$. Thus, $\bar{X}$ is indeed a basis of $\operatorname{Fix}\Phi$.

In order to see (ii), we observe that $\partial^\kappa \bar{x} \in \operatorname{Fix}\Phi$. Thus, (70) follows immediately from (i). Finally, by Proposition 7.3 the pair $(\bar{X}, \bar{\kappa})$ is a Lefschetz complex and clearly $C(\bar{X}) = \operatorname{Fix}\Phi$, which proves (iii). $\square$

It follows from Proposition 9.10 that the map $X^c \ni x \mapsto \bar{x} \in \bar{X}$ is a bijection. Hence, also the map $\mathcal{C} \ni \{x\} \mapsto \bar{x} \in \bar{X}$ is a bijection, which lets us carry over the partial order $\leq_\mathcal{V}$ from $\mathcal{C} \subset \mathcal{V}$ to $\bar{X}$. Thus, for $\bar{x}, \bar{y} \in \bar{X}$ we write $\bar{x} \leq_\mathcal{V} \bar{y}$ if $\{x\} \leq_\mathcal{V} \{y\}$. Then we have the following result.

Proposition 9.12. *Assume that $\mathcal{V}$ is a gradient vector field on a regular Lefschetz complex X. Then the partial order $\leq_\mathcal{V}$ in $\bar{X}$ is a natural partial order on the Lefschetz complex $(\bar{X}, \bar{\kappa})$.*

Proof: First we will prove that the order $\leq_\mathcal{V}$ in $\bar{X}$ is admissible, that is, $\bar{y} \leq_{\bar{\kappa}} \bar{x}$ for $x, y \in X^c$ implies $\bar{y} \leq_\mathcal{V} \bar{x}$. Since the partial order $\leq_{\bar{\kappa}}$ is the transitive closure of $\prec_{\bar{\kappa}}$, it suffices to prove that the inequality $\bar{y} \prec_{\bar{\kappa}} \bar{x}$ implies $\bar{y} \leq_\mathcal{V} \bar{x}$. Thus, assume that $x, y \in X^c$ and $\bar{y} \prec_{\bar{\kappa}} \bar{x}$. Then, $\bar{\kappa}(\bar{x}, \bar{y}) \neq 0$. By Proposition 9.11(i) we have $\partial \bar{x} = \sum_{z \in X^c} a_{xz} \bar{z}$, where $a_{xz} := \bar{\kappa}(\bar{x}, \bar{z})$. By Proposition 9.10 we have $\bar{z} = \Phi^\infty z = z + r_z$, where $|r_z| \subset X^+ \cap |\,[z]^{<_\mathcal{V}}\,|$. Since $y \in X^c \subset X \setminus X^+$, we have

$$\begin{aligned}\langle \partial \bar{x}, y\rangle = \sum_{z \in X^c} a_{xz} \langle \bar{z}, y\rangle &= \sum_{z \in X^c} a_{xz} \left(\langle z, y\rangle + \langle r_z, y\rangle\right) \\ &= \sum_{z \in X^c} a_{xz} \langle z, y\rangle = a_{xy} = \bar{\kappa}(\bar{x}, \bar{y}) \neq 0.\end{aligned}$$

It follows that

$$y \in |\partial \bar{x}| = |\partial \Phi^\infty x| = |\Phi^\infty \partial x| = \left|\Phi^\infty \left(\sum_{w \in |\partial x|} \kappa(x, w) w\right)\right| \subset \bigcup_{w \in |\partial x|} |\Phi^\infty w|.$$

Thus, $y \in |\Phi^\infty u|$ for some $u \in X$ such that $\kappa(x,u) \neq 0$. From Proposition 9.9 we get $[y] \leq_\mathcal{V} [u]$. Since $\kappa(x,u) \neq 0$ we further have $u \in \operatorname{cl} x$, which in turn implies $u \in [u] \cap \operatorname{cl}[x]$. In consequence $[u] \leq_\mathcal{V} [x]$ and $\{y\} = [y] \leq_\mathcal{V} [x] = \{x\}$. Thus, by the definition of $\leq_\mathcal{V}$ in $\bar{X}$ we get $\bar{y} \leq_\mathcal{V} \bar{x}$.

To see that $\leq_\mathcal{V}$ is natural, let $\bar{x}, \bar{y} \in \bar{X}$ satisfy $\bar{x} \leq_\mathcal{V} \bar{y}$ and $\dim \bar{x} = \dim \bar{y}$. Then $\{x\} \leq_\mathcal{V} \{y\}$ and $\dim x = \dim y$. Thus, we cannot have $\{x\} <_\mathcal{V} \{y\}$, because then, by Proposition 9.2(ii), $\dim x = \dim\{x\} < \dim\{y\} = \dim y$. It follows that $\{x\} = \{y\}$ and $\bar{x} = \bar{y}$, which proves that the order is natural. □

In the sequel we consider $\bar{X}$ as a poset ordered by the natural partial order $\leq_\mathcal{V}$ in $\mathcal{V}$, transferred to $\bar{X}$ via $\{x\} \mapsto \bar{x}$, and the triple $(\bar{X}, C(\bar{X}), \partial^{\bar{\kappa}})$ as a natural filtration of $\bar{X}$.

Proposition 9.13. *Assume that $\mathcal{V}$ is a gradient vector field on a regular Lefschetz complex X. Then we have a well-defined filtered morphism*

$$(\mathrm{id}_\mathcal{V}, \Phi) : (\mathcal{V}, C(X), \partial^\kappa) \to (\mathcal{V}, C(X), \partial^\kappa).$$

Moreover, $(\mathrm{id}_\mathcal{V}, \Phi)^n = (\mathrm{id}_\mathcal{V}, \Phi^n)$ is filtered chain homotopic to the identity morphism $\mathrm{id}_{(\mathcal{V},C(X))}$ for every $n \in \mathbb{N}$. In particular, $(\mathrm{id}_\mathcal{V}, \Phi^\infty)$ is filtered chain homotopic to $\mathrm{id}_{(\mathcal{V},C(X))}$.

Proof: Clearly, Φ is a chain map and $\mathrm{id}_\mathcal{V} : (\mathcal{V}, \mathcal{V}_\star) \to (\mathcal{V}, \mathcal{V}_\star)$ is a morphism in DPSet. Hence, to prove that $(\mathrm{id}_\mathcal{V}, \Phi)$ is a well-defined filtered morphism we only have to show that Φ is $\mathrm{id}_\mathcal{V}$-filtered. We will do so by checking property (11) of Corollary 4.6. Consider the down set $\mathcal{L} \in \operatorname{Down}(\mathcal{V})$ and set $L := |\mathcal{L}|$. Then $C(X)_\mathcal{L} = \bigoplus_{V \in \mathcal{L}} C(V) = C(L)$, and we need to verify that $\Phi(C(L)) \subset C(L)$. But this follows from Proposition 9.7, because L is immediately seen to be $\mathcal{V}$-compatible and closed by Proposition 7.9(i).

Since by Proposition 5.14 filtered chain homotopy between morphisms is preserved by composition, in order to prove that $(\mathrm{id}_\mathcal{V}, \Phi)^n = (\mathrm{id}_\mathcal{V}, \Phi^n)$ is filtered homotopic to the identity morphism it suffices to prove that $(\mathrm{id}_\mathcal{V}, \Phi)$ is filtered homotopic to the identity morphism. By the definition of Φ we have $\Phi - \mathrm{id}_{C(X)} = \partial\Gamma + \Gamma\partial$. Thus, we only need to check that Γ is $\mathrm{id}_\mathcal{V}$-filtered. Hence, assume that $\Gamma_{VW} \neq 0$ for some $V, W \in \mathcal{V}$. Then, there exists a $w \in W$ such that $\pi_V(\Gamma w) \neq 0$. In particular, one has $\Gamma w \neq 0$, which implies both $w = w^- \neq w^+$ and $\Gamma w = -\kappa(w^+, w^-)^{-1} w^+$. It follows that $w^+ \in V$. Thus $V = W = \mathrm{id}_\mathcal{V}(W)$, which proves that Γ is $\mathrm{id}_\mathcal{V}$-filtered, in fact, even graded. Finally, since $\Phi^\infty = \Phi^n$ for large $n \in \mathbb{N}$, it follows that also $(\mathrm{id}_\mathcal{V}, \Phi^\infty)$ is filtered chain homotopic to $\mathrm{id}_{(\mathcal{V},C(X))}$. □

Proposition 9.14. *In the situation of the last proposition, we have a well-defined filtered morphism*

$$(\alpha, \varphi) : (\mathcal{V}, C(X), \partial^\kappa) \to (\bar{X}, C(\bar{X}), \partial^{\bar{\kappa}})$$

with $\alpha : \bar{X} \to \mathcal{V}$ defined for $x \in X^c$ by $\alpha(\bar{x}) := \{x\}$ and $\varphi : C(X) \to C(\bar{X})$ defined for $c \in C(X)$ by $\varphi c := \Phi^\infty c$.

Proof: First observe that by Proposition 9.2(iii) we have $\mathcal{V}_\star = \mathcal{C}$, and by Theorem 7.16(i) we have $\bar{X}_\star = \bar{X}$. Therefore, α is a morphism in DSet and since we consider $\bar{X}$ as ordered by the natural order $\leq_\mathcal{V}$, the map α is trivially order preserving, hence also a morphism in DPSet. Obviously, the map φ is a chain map. We will show that φ is α-filtered by checking property (10) of Proposition 4.5. Let $\mathcal{L} \in \operatorname{Down}(\mathcal{V})$ and let $L := |\mathcal{L}|$. Then we have $C(X)_\mathcal{L} = \bigoplus_{V\in\mathcal{L}} C(V) = C(L)$. Observe that $\alpha^{-1}(\mathcal{L})$ is a down set in $\bar{X}$. Indeed, if $\bar{x} \in \alpha^{-1}(\mathcal{L})$ and $\bar{y} \leq_\mathcal{V} \bar{x}$, then $\{y\} \leq_\mathcal{V} \{x\} = \alpha(\bar{x}) \in \mathcal{L}$, which implies $\alpha(\bar{y}) = \{y\} \in \mathcal{L}$ and $\bar{y} \in \alpha^{-1}(\mathcal{L})$. Hence, $\alpha^{-1}(\mathcal{L})^{\leq} = \alpha^{-1}(\mathcal{L})$. It follows that $C(\bar{X})_{\alpha^{-1}(\mathcal{L})^{\leq}} = C(\bar{X})_{\alpha^{-1}(\mathcal{L})} = \bigoplus_{\{u\}\in\mathcal{L}} R\bar{u}$. Thus, in our case, we have to verify condition (10) in the form

$$\Phi^\infty(C(L)) \subset \bigoplus_{\{u\}\in\mathcal{L}} R\bar{u}. \tag{71}$$

In order to verify (71) take a $c \in C(L)$, and let $\bar{c} := \Phi c = \Phi^\infty c$. In view of Proposition 9.10 we have

$$\bar{c} = \sum_{u\in X^c} a_u\bar{u} = \sum_{u\in X^c} a_u u + \sum_{u\in X^c} a_u r_u \tag{72}$$

for some coefficients $a_u \in R$ and chains r_u satisfying $|r_u| \cap X^c = \varnothing$, as well as

$$|r_u| \subset |[u]^{<_\mathcal{V}}|. \tag{73}$$

Consider a $u \in X^c$ such that $\{u\} \notin \mathcal{L}$, i.e., one has $u \notin L$. Since by Proposition 9.7 we have $\varphi(C(L)) = \Phi^\infty(C(L)) \subset C(L)$, it then follows that the identity $\langle \bar{c}, u\rangle = 0$ holds. But now equation (72) gives $\langle \bar{c}, u\rangle = a_u$, because (73) implies $\langle r_u, u\rangle = 0$. Therefore, we have $a_u = 0$ for $u \notin L$, which subsequently proves

$$\bar{c} = \sum_{u\in L} a_u\bar{u} \in \bigoplus_{u\in L} R\bar{u}.$$

This verifies (71) and completes the proof. □

Proposition 9.15. *In the situation of the last proposition, we have a well-defined filtered morphism*

$$(\beta,\psi) : (\bar{X}, C(\bar{X}), \partial^{\bar{\kappa}}) \to (\mathcal{V}, C(X), \partial^\kappa)$$

with $\beta : \mathcal{V} \nrightarrow \bar{X}$ *defined for* $x \in X^c$ *by* $\beta(\{x\}) := \bar{x}$ *and* $\psi : C(\bar{X}) \to C(X)$ *defined for* $c \in C(X)$ *by* $\psi c := c$.

Proof: As we already mentioned, Proposition 9.2 implies that $\mathcal{V}_\star = \mathcal{C}$, and Theorem 7.16(i) gives $\bar{X}_\star = \bar{X}$. Therefore, β is a morphism in DSet. Moreover, since we consider $\bar{X}$ as ordered by the natural order $\leq_\mathcal{V}$, the map β is trivially order preserving, hence also a morphism in DPSet. Obviously, the map ψ is a chain map. We will show that ψ is β-filtered by checking again property (10) of Proposition 4.5. Consider a down set $A \in \operatorname{Down}(\bar{X})$. Clearly, we have $C(\bar{X})_A = C(A)$. If we define $\mathcal{L} := \beta^{-1}(A)$, then in general $\mathcal{L}$

is not a down set in $\mathcal{V}$, but $\mathcal{L}^{\leq \mathcal{V}}$ is. We therefore consider $L := |\mathcal{L}^{\leq \mathcal{V}}|$, and obtain the identity $C(X)_{\beta^{-1}(A)^{\leq \mathcal{V}}} = C(X)_{\mathcal{L}^{\leq \mathcal{V}}} = C(L)$. Thus, in our case, condition (10) has to be verified in the form

$$\psi(C(A)) = C(A) \subset C(L). \tag{74}$$

Consider first a chain $\bar{x} \in A$ for an $x \in X^c$. Then $\beta(\{x\}) = \bar{x} \in A$, which means that $\{x\} \in \beta^{-1}(A) = \mathcal{L}$ and $|x| = \{x\} = [x] \in \mathcal{L} \subset \mathcal{L}^{\leq \mathcal{V}}$. It follows from Proposition 9.10 that $\bar{x} = \Phi^\infty x = x + r_x$ where $|r_x| \subset X^+ \cap |\,[x]^{<\mathcal{V}}\,|$. Hence, the inclusion $|\bar{x}| \subset |x| \cup |r_x| \subset [x] \cup |[x]^{<\mathcal{V}}| = |\,[x]^{\leq \mathcal{V}}\,| \subset |\mathcal{L}^{\leq \mathcal{V}}| = L$ is satisfied, because $[x] \in \mathcal{L} \subset \mathcal{L}^{\leq \mathcal{V}}$ and $\mathcal{L}^{\leq \mathcal{V}}$ is a down set. This immediately shows that $\bar{x} \in C(L)$. Since every chain in $C(A)$ is a linear combination of chains in A, the inclusion (74) follows. □

Theorem 9.16. *Assume that $\mathcal{V}$ is a gradient vector field on a regular Lefschetz complex X. Then the filtered morphisms (α, φ) and (β, ψ) defined in the last two propositions are mutually inverse filtered chain equivalences. In particular, the filtered chain complexes $(\mathcal{V}, C(X), \partial^\kappa)$ and $(\bar{X}, C(\bar{X}), \partial^{\bar{\kappa}})$ are filtered chain homotopic.*

Proof: Clearly, one has $\beta\alpha = \mathrm{id}_{\bar{X}}$. Moreover, if $c \in \operatorname{Fix}\Phi$, then $\Phi^\infty c = c$. Therefore, the identity $\varphi\psi = \mathrm{id}_{\operatorname{Fix}\Phi} = \mathrm{id}_{C(\bar{X})}$ holds and we get

$$(\alpha, \varphi) \circ (\beta, \psi) = (\beta\alpha, \varphi\psi) = (\mathrm{id}_{\bar{X}}, \mathrm{id}_{C(\bar{X})}) = \mathrm{id}_{(\bar{X}, C(\bar{X}))}.$$

Thus, $(\alpha, \varphi) \circ (\beta, \psi)$ is filtered homotopic to $\mathrm{id}_{(\mathcal{C}, C(\bar{X}))}$ via the zero filtered homotopy. In the opposite direction we have

$$(\beta, \psi) \circ (\alpha, \varphi) = (\alpha\beta, \psi\varphi) = (\mathrm{id}_{\mathcal{V}|\mathcal{C}}, \Phi^\infty). \tag{75}$$

Since $\mathcal{V}_\star = \mathcal{C}$, we see that $(\mathrm{id}_{\mathcal{V}|\mathcal{C}}, \Phi^\infty) \sim_e (\mathrm{id}_{\mathcal{V}}, \Phi^\infty)$. Hence, it follows from Proposition 9.13 and (75) that $(\alpha, \varphi) \circ (\beta, \psi)$ is filtered chain homotopic to the identity $\mathrm{id}_{(\mathcal{V}, C(X))}$. The conclusion follows. □

Theorem 9.17. *Let $\mathcal{V}$ be a gradient combinatorial vector field on a regular Lefschetz complex X. Then $\mathcal{C}$, the collection of the critical vectors of $\mathcal{V}$, is a Morse decomposition of $\mathcal{V}$. It has exactly one connection matrix which coincides with the $(\bar{X}, \bar{X})$-matrix of $\partial^\kappa_{|\operatorname{Fix}\Phi}$ up to a graded similarity.*

Proof: It follows from Proposition 9.1 that $\mathcal{C}$ is a Morse decomposition of $\mathcal{V}$ and the partition induced on X by this Morse decomposition is $\mathcal{V}$. Thus, by Definition 8.8, the Conley complex of $\mathcal{C}$ is the Conley complex of $(\mathcal{V}, C(X), \partial^\kappa)$. From Corollary 6.12 and Theorem 9.16 we obtain that the Conley complex of $(\mathcal{V}, C(X), \partial^\kappa)$ is isomorphic in PFCC to the Conley complex of $(\bar{X}, C(\bar{X}), \partial^{\bar{\kappa}})$ with $\bar{X}$ ordered by a natural partial order. Since, by Proposition 9.11 and Definition 7.15, the latter is the Conley complex of a natural filtration of a Lefschetz complex, the conclusion follows directly from Theorem 7.16(ii). □

Example 9.18. Consider the partition $\mathcal{E}_1 := \mathcal{E}_{\mathcal{M}_1}$ associated with the Morse decomposition $\mathcal{M}_1$ of the vector field $\mathcal{V}_1$ in the middle of Figure 5. In

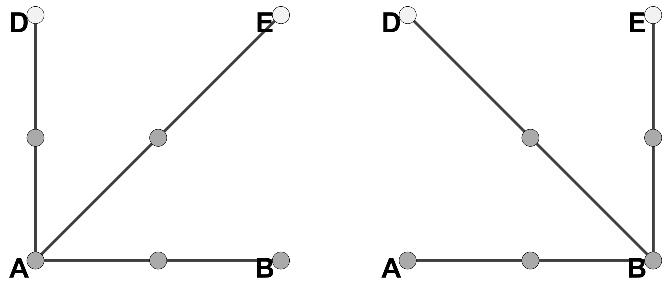

FIGURE 6. Conley complexes $(\bar{X}, \partial^{\bar{\kappa}})$ for the Morse decompositions $\mathcal{M}_1$ (left) and $\mathcal{M}_2$ (right) of the combinatorial gradient vector fields $\mathcal{V}_1$ and $\mathcal{V}_2$ in Figure 5, visualized as κ-subcomplexes of simplicial complexes.

this simple case it coincides with the vector field $\mathcal{V}_1$. Hence, the Hasse diagram of the partial order $\leq_{\mathcal{E}_1}$ coincides with diagram (56). It follows that $\mathcal{V}_1$ is a gradient vector field and by Theorem 9.17 $\mathcal{M}_1$ has exactly one connection matrix. One can verify that the morphisms (α', h') and $((\alpha')^{-1}, g')$ constructed as in Example 6.13 are also filtered with respect to the filtration $(\mathcal{E}_1, C(X), \partial^\kappa)$. Therefore, arguing as in Example 6.13 we conclude that the Conley complex of $\mathcal{M}_1$ is the filtered chain complex in Example 5.9 with the associated connection matrix (27). In fact, the homomorphism h' coincides with homomorphism $\Phi_{\mathcal{V}_1}$ and $\Phi_{\mathcal{V}_1} = \Phi^\infty_{\mathcal{V}_1}$ in this case. An analogous argument shows that the Conley complex of $\mathcal{M}_2$ is the filtered chain complex in Example 6.16 with the associated connection matrix (31). Both Conley complexes are visualized in Figure 6 as κ-subcomplexes of simplicial complexes.

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